Phase transition in multicomponent field theory at finite temperature

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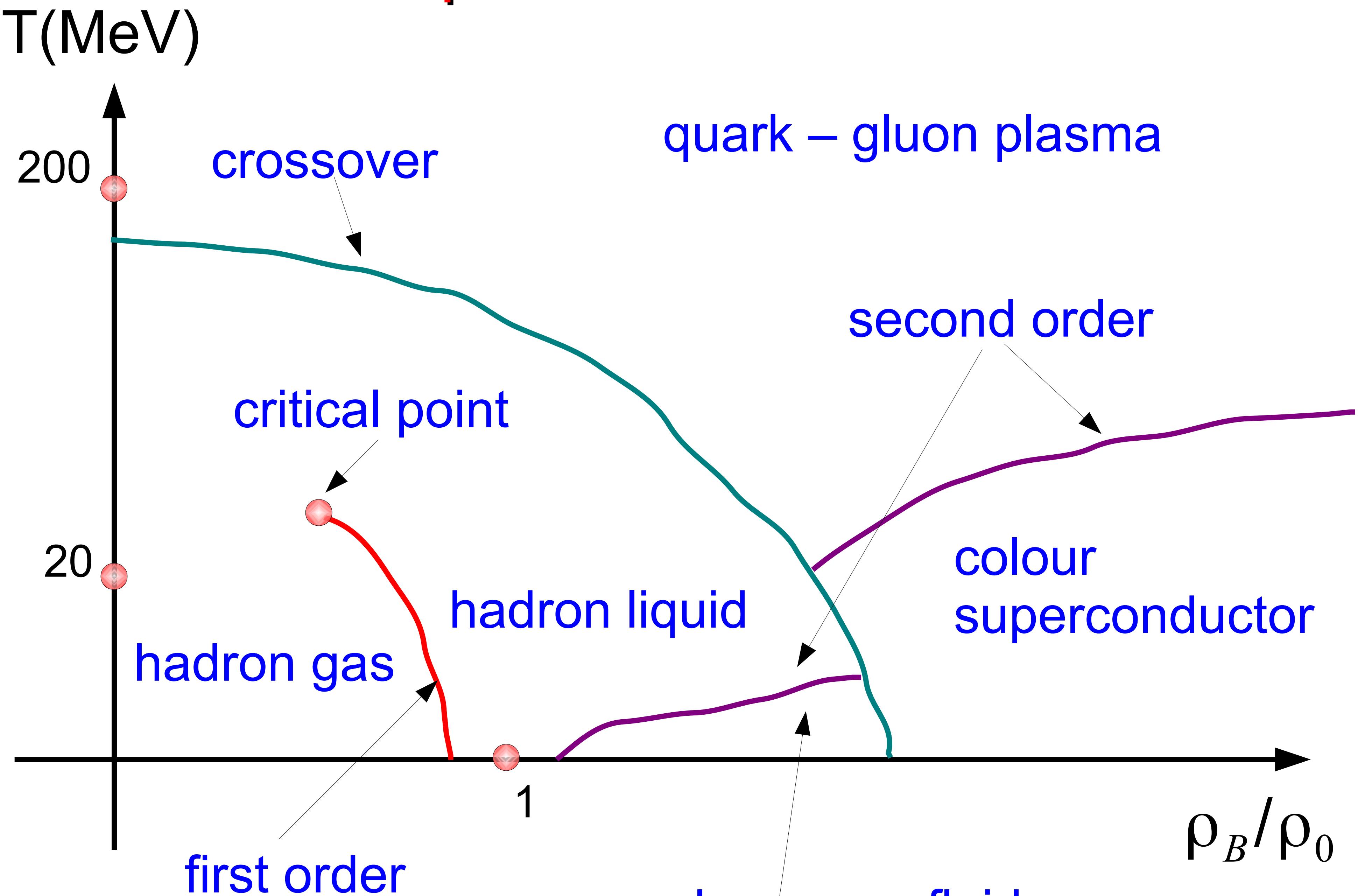
Possible phases of nuclear matter



Problems

- Description of phase transitions
- Microscopic theory
- Perturbation theory
- Divergent series
- Effective limit ?

Divergent series

$$\rho(x) : \mathbb{R} \to \mathbb{R}$$

Perturbation theory

$$\rho(x) \simeq \rho_k(x) \qquad (x \to 0)$$

$$\rho_k(x) = \rho_0(x)\left(1 + \sum_{n=1}^{k} a_n x^n\right)$$

$$\rho_k(x) \to ? \qquad (k \to \infty)$$

x can be not small

Divergent for any $x \neq 0$

Normalization

$$f_k(x) \equiv \frac{\rho_k(x)}{\rho_0(x)}$$

$$f_k(x) = 1 + \sum_{n=1}^{k} a_n x^n$$

$$\lim_{x \to 0} f_k(x) = 1$$

Optimized perturbation theory

Control functions

$$u_k = u_k(x)$$

$$f_k(x) \to F_k(x, u_k)$$

Convergent sequence

$$\{ F_k(x, u_k(x)) \}$$

Cauchy criterion

For each ε, there exists k_ε, so that

$$| F_{k+p}(x, u_{k+p}) - F_k(x, u_k) | < \varepsilon$$

for

$$k \geq k_\varepsilon, \quad p \geq 1$$

Introduction of control functions

1. Through initial conditions
2. Through change of variables
3. Through function transformation

Initial conditions

Perturbation theory

$$H = H_0(u) + \varepsilon[H - H_0(u)]$$

$$\varepsilon \to 1$$

$$\psi(x,u), \qquad G_k(x,u)$$

$$F_k(x,u) = \langle \hat{A}(x) \rangle_k$$

Iterative procedure

$$\hat{B} f(x) = 0$$

Start with

$$F_0(x,u)$$

$$F_{k+1}(x,u_{k+1}) = (1 + \hat{B}) F_k(x,u_k)$$

Change of variables

$$x = x_k(z, u_k), \qquad z = z_k(x, u_k)$$

Substitution of x

$$f_k(x) = f_k(\, x_k(z, u_k)\,)$$

Expansion in z

$$f_k(\, x_k(z, u_k)\,) \rightarrow \overline{f_k}(z, u_k)$$

Substitution of z

$$\overline{f_k}(z, u_k) = \overline{f_k}(\, z_k(x, u_k), u_k\,)$$

$$F_k(x, u_k) = \overline{f_k}(\, z_k(x, u_k), u_k\,)$$

Function transformation

$$\hat{T}(u)\,f(x) = F(x,u)$$

$$f(x) = \hat{T}(u)\,F(x,u)$$

$$F_k(x,u_k) = \hat{T}^{-1}(u_k)\,f_k(x)$$

Cauchy cost functional

$$C[u] = \frac{1}{2}\sum_{n=0}^{\infty} | F_{n+1}(x, u_{n+1}) - F_n(x, u_n) |^2$$

$$\min | F_{k+1}(x, u_{k+1}) - F_k(x, u_k) |$$

Euler discretization $F_{k+1}(x, u_{k+1}) - F_k(x, u_k) \rightarrow$

$$F_{k+1}(x, u_k) - F_k(x, u_k) + \\ + \frac{\delta F_k(x, u_k)}{\delta u_k} (u_{k+1} - u_k)$$

Optimization

$$\min \left| F_{k+1}(x, u_k) - F_k(x, u_k) + \right.$$

$$\left. + \frac{\delta F_k(x, u_k)}{\delta u_k} \left(u_{k+1} - u_k \right) \right|$$

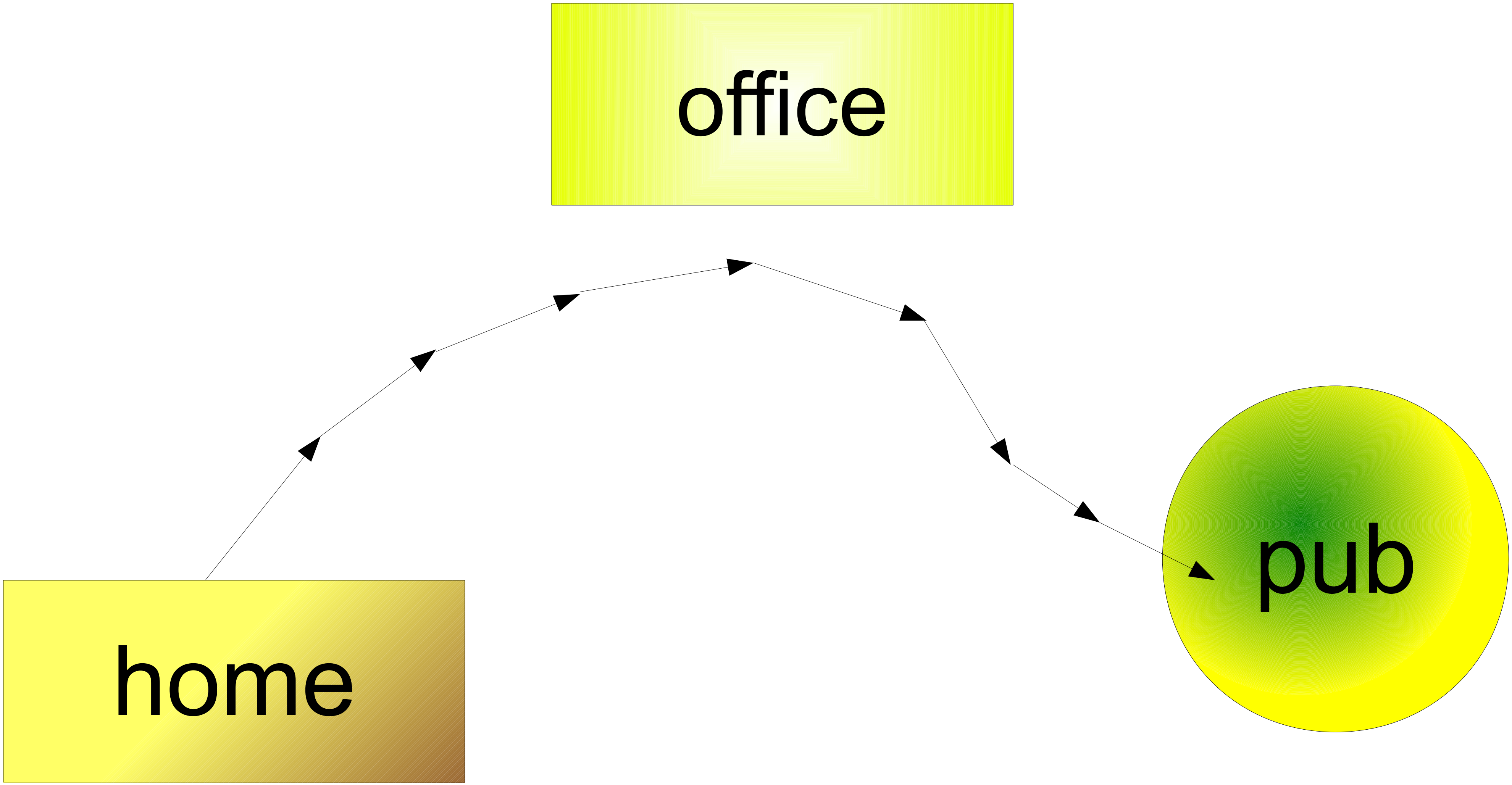
u_k weakly varying, difference condition

$$F_{k+1}(x, u_k) - F_k(x, u_k) = 0$$

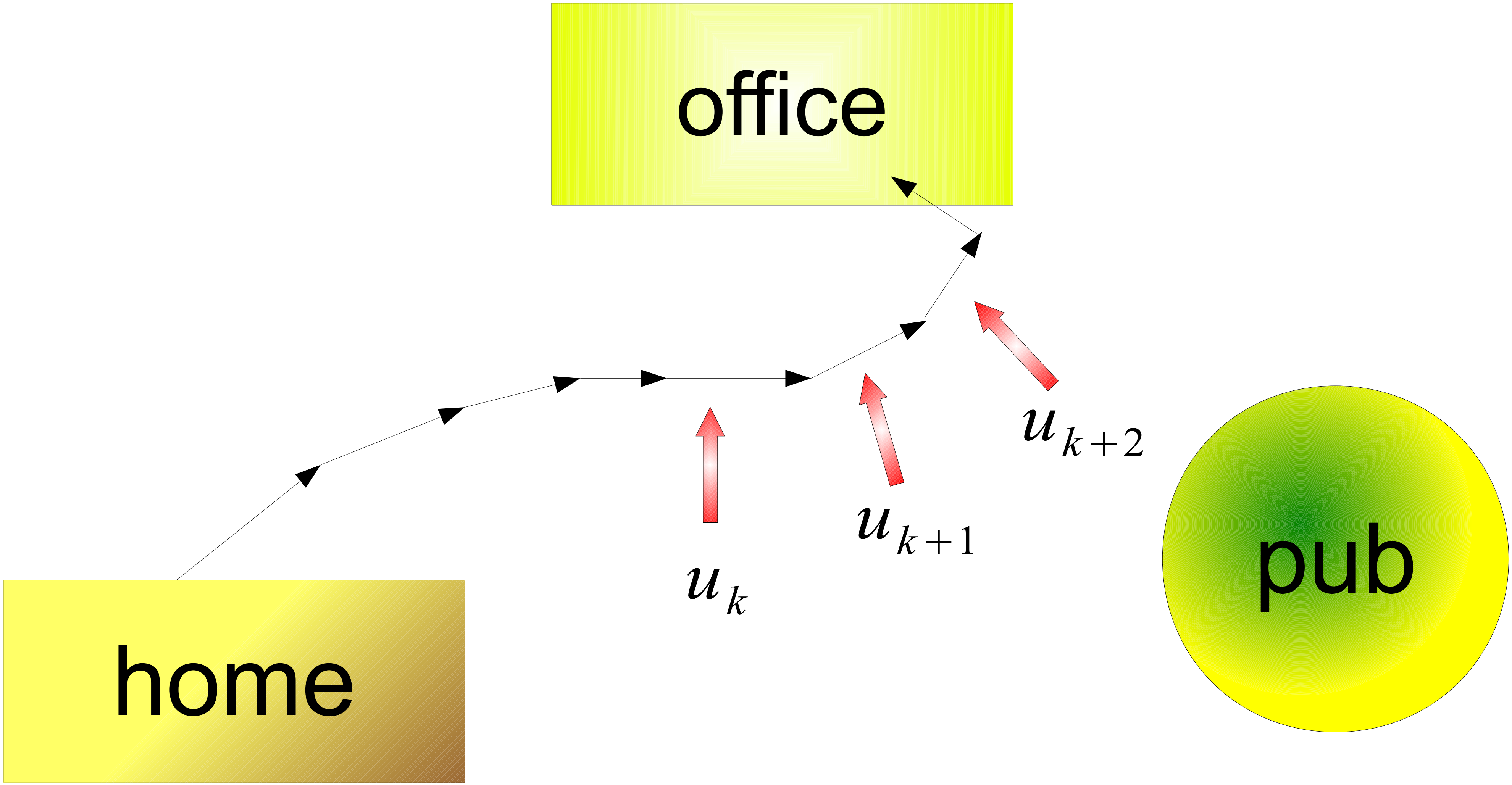
F_k weakly varying, differential condition

$$\frac{\delta F_k(x, u_k)}{\delta u_k} = 0$$

Divergence



Control functions



Optimized perturbation theory

$\{ f_k(x) \}$ divergent

Control functions $u_k(x)$

$$f_k(x) \longleftrightarrow F_k(x, u_k(x))$$

$\{ F_k(x, u_k(x)) \}$ convergent

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Questions

1. How to improve accuracy, with a given number of perturbative terms?
2. How to choose initial conditions, if several are admissible?
3. How to control procedure stability, when no exact solutions are available?

Self – similar approximation theory

Approximation order k ⟷ discrete time

$$\{ F_k(x, u_k) \} \longleftrightarrow \{ y_k(\varphi) \}$$

Approximation sequence ⟷ cascade trajectory

Sequence limit ⟷ fixed point

Control of convergence ⟷ stability of dynamics

Approximation cascade

Reonomic constraint

$$F_0(\, x\,, u_k(x)\,) \;=\; \varphi\,, \qquad x \;=\; x_k(\varphi)$$

Expansion function

$$x_k(\varphi)$$

Endomorphism

$$y_k(\varphi) \;\equiv\; F_k(\, x_k(\varphi), u_k(x_k(\varphi))\,)$$

Cascade

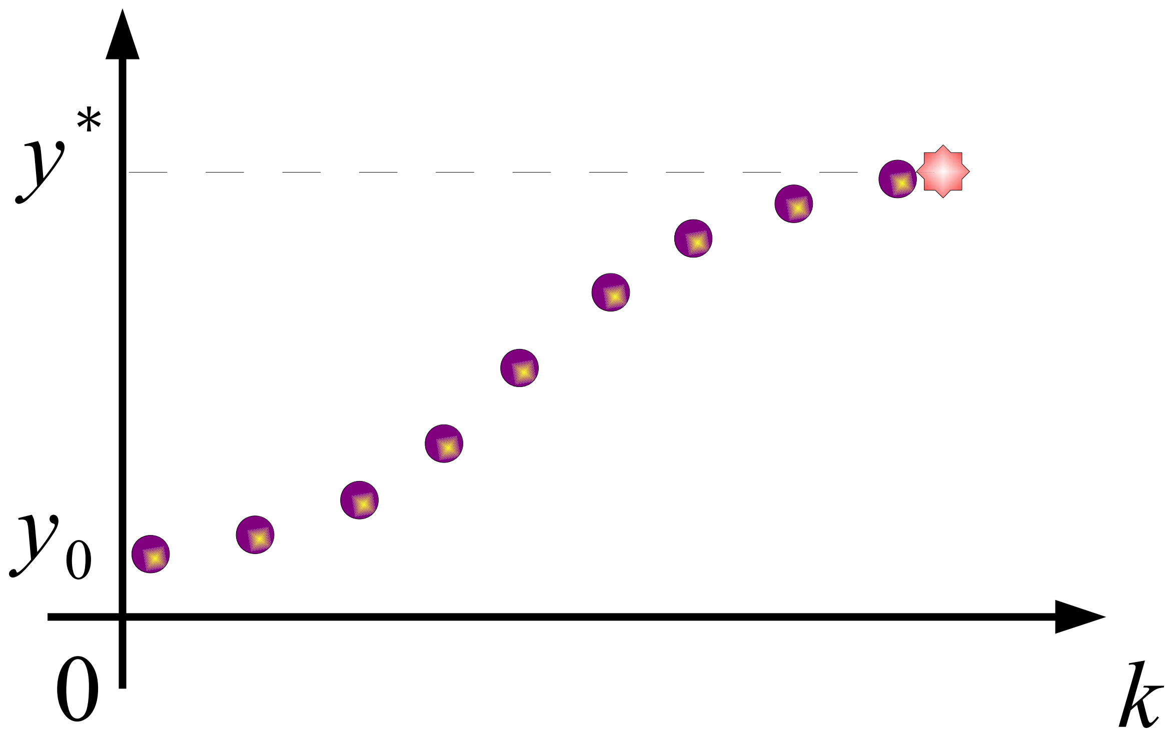
$$\{\, y_k(\varphi) : k = 0\,, 1\,, 2\,, \ldots \,\}$$

Initial condition

$$y_0(\varphi) \;=\; \varphi$$

Fixed point

$$\{\, y_k(\varphi) \,\} \longleftrightarrow \{\, F_k(x, u_k) \,\}$$



$$y_k(y^*(\varphi)) = y^*(\varphi)$$

$y^*(\varphi)$ effective limit of approximation sequence

Group self – similarity

$$y_{k+p}(\varphi) = y_k(\, y_p(\varphi)\,)$$

Identity for

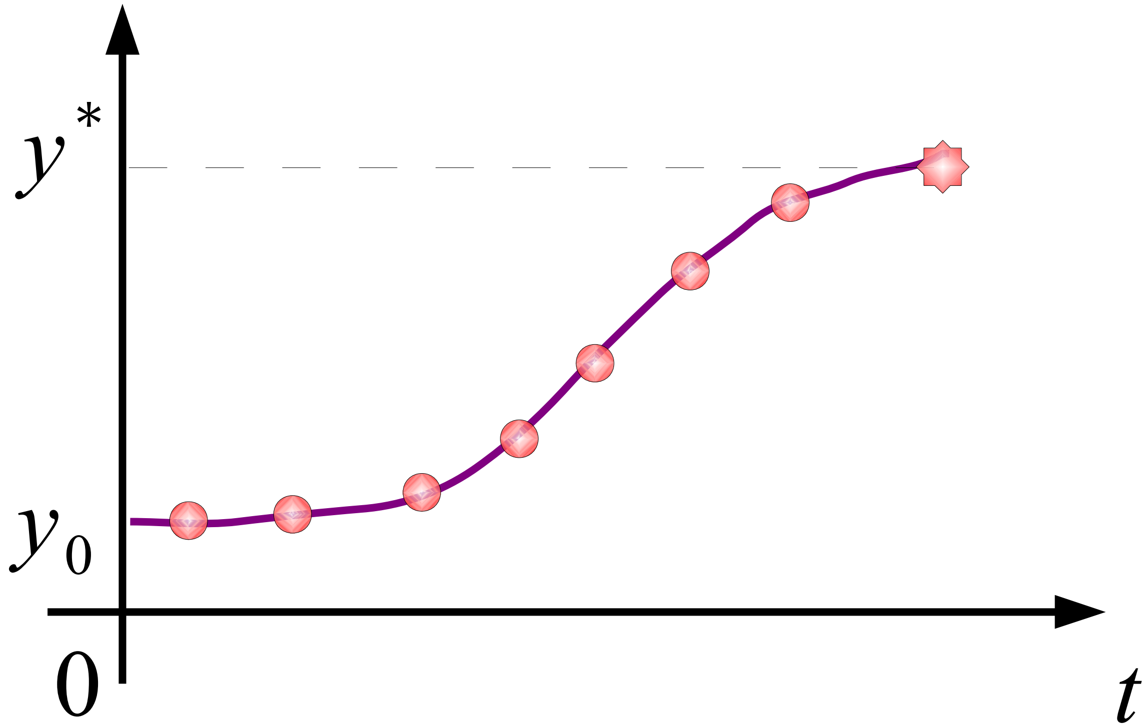
$$\varphi \to y^*(\varphi)$$

Semigroup

$$y_{k+p} = y_k \cdot y_p, \qquad y_0 = 1$$

Embedding into flow

$$\{\, y_k(\varphi):\ k \in \mathbb{Z}_+ \,\} \in \{\, y_t(\varphi):\ t \in \mathbb{R}_+ \,\}$$



Lie equation

$$\frac{\partial}{\partial t}\, y_t(\varphi) = v(\, y_t(\varphi)\,)$$

Evolution integral

$$\int_{y_k}^{y_k^*} \frac{dy}{v_k(y)} = \tau_k$$

Control time

$$\tau_k$$

Euler discretization

$$v_k(\varphi) = F_{k+1}(x_k, u_k) - F_k(x_k, u_k) +$$
$$+ (u_{k+1} - u_k) \frac{\partial}{\partial u_k} F_k(x_k, u_k)$$

Cascade velocity

Accuracy improvement

$$\int_{F_k}^{f_k^*} \frac{d\varphi}{v_k(\varphi)} = \tau_k$$

$$f_k^*(x) \equiv F_k^*(x, u_k(x))$$

$$v_k \to 0, \qquad f_k^* \to F_k$$

$$\min |v_k| \longleftrightarrow \min C[u]$$

Minimal velocity condition $\rightarrow u_k(x)$

$$\min |v_k| \neq 0$$

Dynamics

$$F_k \to f_k^*$$

Stability conditions

Local multiplier

$$\mu_k(\varphi) \equiv \frac{\partial}{\partial \varphi} y_k(\varphi)$$

Local stability

$$|\mu_k(\varphi)| < 1$$

Multiplier at fixed point

$$\mu_k^*(\varphi) \equiv \mu_k(y_k^*(\varphi))$$

Stable fixed point

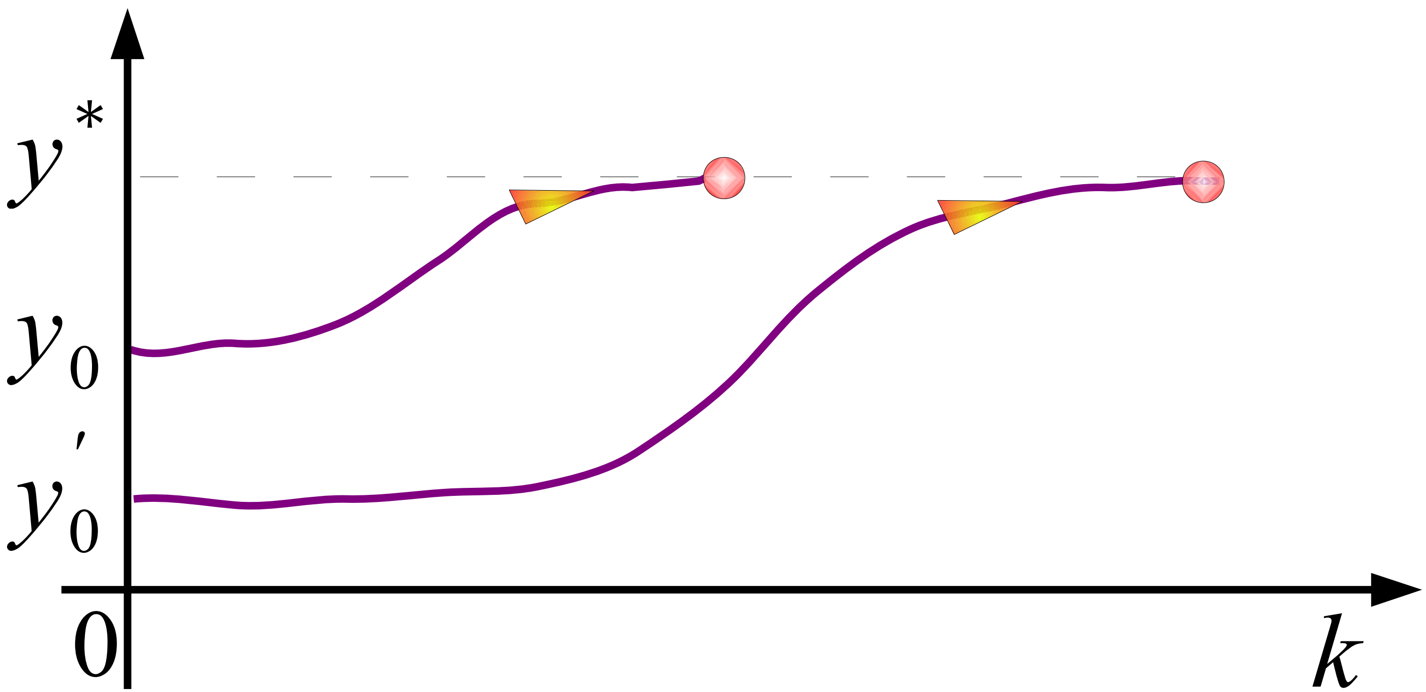
$$|\mu_k^*(\varphi)| < 1$$

Maximal multiplier at fixed point

$$\mu_k^* = \sup_\varphi |\mu_k^*(\varphi)| = \sup_x |\mu_k(f_k^*(x))|$$

Uniformly stable fixed point $|\mu_k| < 1$

Minimal of maximal multiplier



Right root approximants

$$f_k^*(x) = ((\ldots(1 + A_1 x)^{n_1} + A_2 x^2)^{n_2} + \ldots A_k x^k)^{n_k}$$

A_i, n_i from behaviour at $x \to \infty$

Left root approximants

$$f_k^*(x) = (((\ldots(1 + A_1 x)^2 + A_2 x^2)^{3/2} + A_3 x^3)^{4/3} + \ldots + A_k x^k)^{n_k}$$

A_i from expansion at $x \to 0$

n_k from behaviour at $x \to \infty$

Continued root approximants

$$f_k^*(x) = (1 + A_1 x (1 + A_2 x \dots (1 + A_k x)^s)^s \dots)^s$$

A_i from expansion at $x \to 0$

s from behaviour at $x \to \infty$

For s = -1, continued fractions, Padé approximants

Exponential approximants

$$f_k^*(x) = \exp(b_1 x \exp(b_2 x \ldots \exp(b_k x)) \ldots)$$

$$b_n = \frac{a_n (1 + a_1^2)}{n\, a_{n-1} (1 + a_n^2)}$$

a_n from expansion at $x \to 0$

Factor approximants

$$f_k^*(x) = \prod_{n=1}^{N_k} (1 + A_i x)^{n_i}$$

$$N_k = \begin{cases} k/2, & k=2,4,... \\ (k+1)/2, & k=3,5,... \end{cases}$$

$A_i, \ n_i$ from expansion at $x \to 0$

N – Component φ^4 field theory

$$H[\varphi] = \int \left\{ \frac{1}{2} \left[\frac{\partial \varphi(x)}{\partial x} \right]^2 + \frac{m^2}{2} \varphi^2(x) + \frac{\lambda}{4!} \varphi^4(x) \right\} dx$$

$$\varphi(x) = \{ \varphi_n(x) : \; n = 1, 2, \ldots, N \}$$

$$x = \{ x_\alpha : \; \alpha = 1, 2, \ldots, d \}$$

$$\varphi^2(x) \equiv \sum_{n=1}^{N} \varphi_n^2(x)$$

$$\left[\frac{\partial \varphi(x)}{\partial x} \right]^2 \equiv \sum_{n=1}^{N} \sum_{\alpha=1}^{d} \left[\frac{\partial \varphi_n(x)}{\partial x_\alpha} \right]^2$$

$O(N)$ symmetry

$$\varphi_n(x) \rightarrow -\varphi_n(x) \qquad (n = 1, 2, \ldots, N)$$

$$H[-\varphi] \rightarrow H[\varphi]$$

$$\langle \varphi \rangle_{\mathcal{H}} = 0$$

Thermodynamical potential

$$F \equiv -T \ln \mathrm{Tr}\, e^{-\beta H[\varphi]}$$

$$F(-\langle \varphi \rangle) = F(\langle \varphi \rangle)$$

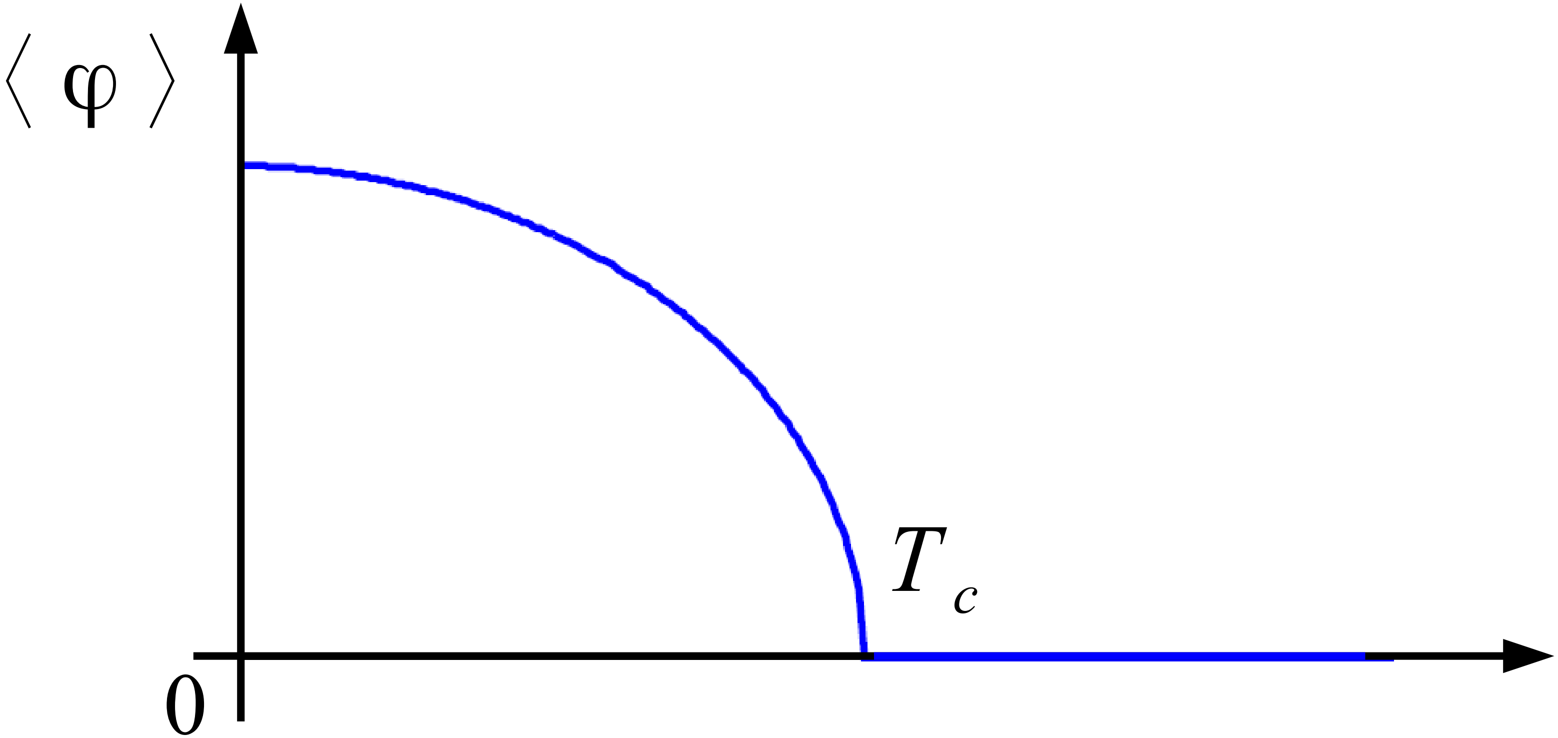
Phase Transition

$$\langle \varphi(x) \rangle = 0 \qquad (T > T_c)$$

Spontaneous symmetry breaking

$$\langle \varphi(x) \rangle \neq 0 \qquad (T < T_c)$$

$$F(\langle \varphi(x) \rangle \neq 0) < F(\langle \varphi \rangle \equiv 0)$$



Critical exponents

Critical region

$$\tau \equiv \frac{|T - T_c|}{T_c} \to 0$$

Specific heat

$$C_V \propto \tau^{-\alpha}$$

Order parameter (function)

$$\langle \varphi \rangle \propto \tau^{\beta}$$

Compressibility

$$\kappa_T \propto \tau^{-\gamma}$$

External field $(T = T_c)$

$$h \propto |\langle \varphi \rangle|^{\delta}$$

Pair correlation function $\left(|\vec{r}| \rightarrow \infty \right)$

$$g(\vec{r}) \propto \frac{\exp(-r/\xi)}{|\vec{r}|^{d-2+\eta}}$$

Correlation length

$$\xi \propto \tau^{-\nu}$$

Vertex $\left(T = T_c \right)$

$$\Gamma(k) \propto 1 + ck^{\omega}$$

Scaling relations

Griffith

$$\alpha + \beta(1+\delta) = 2$$

Widom

$$\gamma + \beta(1-\delta) = 0$$

Rushbrook

$$\alpha + 2\beta + \gamma = 2$$

Hyperscaling relations

$$\alpha = 2 - \nu d$$

$$\beta = (d - 2 + \eta)\frac{\nu}{2}$$

$$\gamma = (2 - \eta)\nu$$

$$\delta = \frac{d + 2 - \eta}{d - 2 + \eta}$$

Independent:

$$\eta, \ \nu, \ \omega$$

Critical exponents

Series from $\varepsilon \equiv 4 - d$ expansion

$$d \to 3, \quad \varepsilon \to 1$$

$$\sum_{n=0}^{5} a_n \varepsilon^n \to \prod_i (1 + A_i \varepsilon)^{n_i}$$

Table

N	α	β	γ	δ	η	ν	ω
-2	0.5	0.25	1	5	0	0.5	0.80118
-1	0.36844	0.27721	1.07713	4.88558	0.019441	0.54385	0.79246
0	0.24005	0.30204	1.15587	4.82691	0.029706	0.58665	0.78832
1	0.11465	0.32509	1.23517	4.79947	0.034578	0.62854	0.78799
2	-0.00625	0.34653	1.31320	4.78962	0.036337	0.66875	0.78924
3	-0.12063	0.36629	1.38805	4.78953	0.036353	0.70688	0.79103
4	-0.22663	0.38425	1.45813	4.79470	0.035430	0.74221	0.79296
5	-0.32290	0.40033	1.52230	4.80254	0.034030	0.77430	0.79492
6	-0.40877	0.41448	1.57982	4.81160	0.032418	0.80292	0.79694
7	-0.48420	0.42676	1.63068	4.82107	0.030739	0.82807	0.79918
8	-0.54969	0.43730	1.67508	4.83049	0.029074	0.84990	0.80184
9	-0.60606	0.44627	1.71352	4.83962	0.027463	0.86869	0.80515
10	-0.65432	0.45386	1.74661	4.84836	0.025928	0.88477	0.80927
50	-0.98766	0.50182	1.98402	4.95364	0.007786	0.99589	0.93176
100	-0.89650	0.48334	1.92981	4.99264	0.001229	0.96550	0.97201
1000	-0.99843	0.49933	1.99662	4.99859	0.000235	0.99843	0.99807
10000	-0.99986	0.49993	1.99966	4.99986	0.000024	0.99984	0.99979
∞	-1	0.5	2	5	0	1	1

Conclusion

- Methods for defining limits of divergent series.
- Simple calculations.
- The results close to other complicated methods, such as Padé – Borel summation.
- In agreement with Monte Carlo simulations.
- For $N = -2$ and $N \to \infty$, exact known values.