Cut Mellin Moments (CMM) approach and generalized DGLAP equations

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Abstract

In collaboration with S.V. Mikhailov and O.V. Teryaev
Based on
D.K. and A. Kotlorz, Phys. Lett. B644 (2007) 284
D.K. and S.V. Mikhailov, JHEP 06 (2014) 065

Generalized Cut Mellin Moments (CMM) obtained by
multiple integrations $(k > 0)$ as well as
multiple differentiations $(k < 0)$
of the original parton distribution also satisfy the DGLAP equations
with the simply transformed evolution kernel: Px^k
The similar generalized evolution equation, with correspondingly modified coefficient functions, can be obtained also for structure functions.
Use appropriate classes of CMM for the available experimental kinematic range enables enhancement of x-region with smaller uncertainties.

CMM approach is a novel tool providing a rich variety
of further possible ways to test QCD

Motivation

PDFs play the central role in DGLAP approach

Moments - a natural candidate in QCD analysis

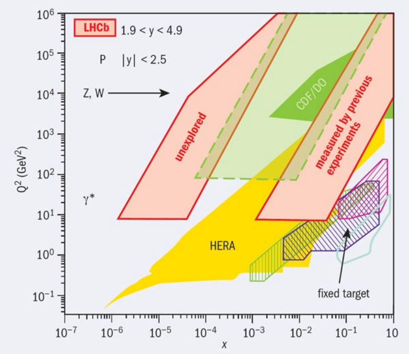
- originate from OPE - basic formalism of the quantum field theory
- directly reffer to sum rules - fundamental relations in QCD
- contributions to momentum or spin of the nucleon coming from quarks and gluons (spin crisis)

Evolution equations for CMM

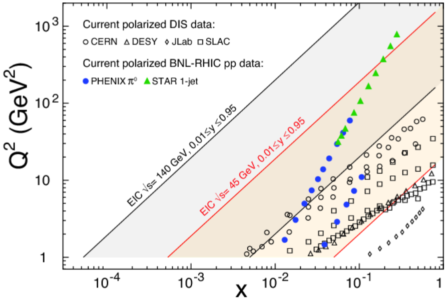
- enables directly to study physical values and their evolution
- allows to avoid uncertainties from non-available experimentally low-x region

Motivation: EXPERIMENTS PROVIDE CUT MOMENTS!

CMM allow one to avoid unphysical $x \to 0$ and gives answer for any x_0
Indirect determination of low-x contributions to BSR



(CERN Courier 2012)



(A.Accardi et al. arXiv:1212.1701)

Contents

Cut Mellin Moments (CMM) - definitions

$$\underbrace{f_n(Q^2) = \int_0^1 dx\, x^{n-1} f(x, Q^2)}_{\text{uncut moment}}$$

$$\underbrace{f_n(x_{min}, Q^2) = \int_{x_{min}}^1 dx\, x^{n-1} f(x, Q^2)}_{\text{cut moment}}$$

$$\underbrace{f_n(x_{min}, x_{max}, Q^2) = \int_{x_{min}}^{x_{max}} dx\, x^{n-1} f(x, Q^2)}_{\text{double cut moment}}$$

Evolution equations for CMM

Main findings (A. Kotlorz and D.K, Phys. Lett. B644 (2007), 284)

$$\frac{df_n^{NS}(x,Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi}\,(P'_{qq}(n) * f_n^{NS})(x,Q^2)$$

$$\frac{d}{d\ln Q^2}\begin{pmatrix} f_n^S(x,Q^2) \\ G_n(x,Q^2)\end{pmatrix} = \frac{\alpha_s(Q^2)}{2\pi}\begin{pmatrix} P'_{qq}(n) & P'_{qG}(n) \\ P'_{Gq}(n) & P'_{GG}(n)\end{pmatrix} * \begin{pmatrix} f_n^S(x,Q^2) \\ G_n(x,Q^2)\end{pmatrix}$$

Rescaled splitting functions:

$$P'_{ij}(n,x) = x^n\, P_{ij}(x)$$

$$P_{ij}(x) = P_{ij}^{(0)}(x) + \frac{\alpha_s(Q^2)}{2\pi}P_{ij}^{(1)}(x) + \left(\frac{\alpha_s(Q^2)}{2\pi}\right)^2 P_{ij}^{(2)}(x) + \cdots$$

Anomalous dimension:

$$\gamma'_{s,n} \equiv \int_0^1 dx\, x^{s-1}\, P'(n,x) = \gamma_{s+n}$$

Comparison of the standard and CMM aproaches

PDF

Input: PDFs $f(x, Q_0^2)$

$\Downarrow$

Evolution from Q_0^2 to Q^2

$$\frac{\partial f(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} (P * f)$$

$\Downarrow$

Results: PDF $f(x, Q^2)$

CMM

Input: CMM $f_n(x_{min}, Q_0^2)$

$\Downarrow$

Evolution from Q_0^2 to Q^2

$$\frac{\partial f_n(x_{min}, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s(Q^2)}{2\pi} (P' * f_n)$$

$\Downarrow$

Results: CMM $f_n(x_{min}, Q^2)$

$$P'(n, z) = z^n P(z)$$

one can use standard methods of solving DGLAP evolution equations
(useful relations between uncut and cut moments, D.K. 2011)

Evolution equations for double CMM

Since the experimental data cover only a limited range of x, except very small $x \to 0$ as well as large $x \to 1$, it is very natural and convenient to deal with the double cut moments. Truncation at large x is less important in comparison to the small-x limit because of the rapid decrease of the parton densities as $x \to 1$, nevertheless a comprehensive analysis requires an equal treatment of the both cut limits.

double CMM is a subtraction of two CMM

$$f_n(x_{min}, x_{max}, Q^2) = f_n(x_{min}, Q^2) - f_n(x_{max}, Q^2)$$

and also satisfies the DGLAP-type evolution with a kernel $P'(n, z)$

Generalization of DGLAP equations (S.V. Mikhailov and D.K., JHEP 06 (2014) 065)

If $f(x, Q^2)$ is a solution of DGLAP equation with the kernel P(y):

$$\dot{f} \equiv \frac{\partial f(z, Q^2)}{\partial \ln Q^2} = (P * f)(z) \equiv \int_0^1 P(y)\, f(x, Q^2)\, \delta(z - xy)\, dx\, dy,$$

then the multi-integrated function which is a generalization of the CMM

$$f(z; n_1, n_2, ..., n_k) = \int_z^1 z_k^{n_k - 1} dz_k \int_{z_k}^1 z_{k-1}^{n_{k-1}-1} dz_{k-1} \ldots \int_{z_2}^1 z_1^{n_1 - 1}\, f(z_1)\, dz_1$$

is also the solution of DGLAP equation:

Generalization of DGLAP equations

$$\dot{f}(z; n_1, n_2, ..., n_k) = (\mathcal{P} * f)(z; n_1, n_2, ..., n_k)$$

with the kernel

$$\mathcal{P}(y) = P(y) \cdot y^{n_1+n_2+...+n_k}$$

Special cases:

For $n_1 = n_2 = ... = n_k = n$ one has

$$f(z; \{n\}_k) = \int_z^1 \left[\frac{t^n - z^n}{n}\right]^{k-1} \frac{f(t)}{(k-1)!} t^{n-1} \, dt \, ,$$

$$\mathcal{P}(y) = P(y) \cdot y^{kn}$$

For $k = 1$ one obtains evolution eq. for cut n-th moment

$$f(z; n) = \int_z^1 t^{n-1} \, f(t) \, dt$$

Generalization of DGLAP equations

Evolution equation at $k = 1$ (2007):

$$\dot{f}(z,n) \equiv \frac{\partial f(z;n,Q^2)}{\partial \ln Q^2} = (\mathcal{P} * f)(z;n,Q^2) \equiv$$

$$\int_0^1 \mathcal{P}(y)\, f(x;n,Q^2)\, \delta(z - xy)\, dx\, dy,$$

where

$$\mathcal{P}(y) = P(y) \cdot y^n$$

If one puts $z = 0$, it reduces to the well known standard renorm-group equation for the moments $f(0;n,Q^2)$:

$$\frac{\partial f(0;n,Q^2)}{\partial \ln Q^2} = \left(\int_0^1 P(y)\, y^{n-1}\, dy \right) \cdot f(0;n,Q^2) \equiv \gamma(n) \cdot f(0;n,Q^2)$$

Generalized CMM and their evolution kernels

$$f(z;n_1,n_2,...,n_k) = \int\limits_z^1 z_k^{n_k-1}dz_k \int\limits_{z_k}^1 z_{k-1}^{n_{k-1}-1}dz_{k-1} \dots \int\limits_{z_2}^1 z_1^{n_1-1}\, f(z_1)\, dz_1$$

$$\mathcal{P}(y) = P(y)\cdot y^{n_1+n_2+\ldots+n_k}$$

$$f(z;n,n,...,n) = \int_z^1 \left[\frac{t^n - z^n}{n}\right]^{k-1} \frac{f(t)}{(k-1)!}\, t^{n-1}\, dt$$

$$\mathcal{P}(y) = P(y)\cdot y^{kn}$$

$$f(z;n,1,...,1) = \int_z^1 (t-z)^{k-1}\, \frac{f(t)}{\Gamma(k)}\, t^{n-1}\, dt$$

$$\mathcal{P}(y) = P(y)\cdot y^{n+k-1}$$

Generalized CMM and their evolution kernels

$$f(z;\underbrace{n,1,...,1}_{k}) = \underbrace{\int\limits_z^1 dz_k \int\limits_{z_k}^1 dz_{k-1} \ldots \int\limits_{z_2}^1}_{k} z_1^{n-1} \, f(z_1)\, dz_1$$

hence the inverse operator:

$$\underbrace{z^{n-1} f(z)}_{\text{ev.kernel:} \quad Py^{n-1}} = \left(-\frac{d}{dz}\right)^k \underbrace{f(z;n,1,...,1)}_{\text{ev.kernel:} \quad Py^{n+k-1}}$$

Generalized evolution for derivatives

$$\left(-\frac{d}{dz}\right)^k [f(z)z^n]$$

$$\mathcal{P}(y) = P(y)\cdot y^{n-k}$$

Summary of Generalized CMM and their Evol. Kernels

	Generalized CMM	DGLAP kernel
1.	$f(x)$	$P(y)$
2.	$x^n f(x)$	$P(y)\cdot y^n$
3.	$\int_z^1 dx\, x^{n-1}\, f(x)$	$P(y)\cdot y^n$
4.	$f(z; n_1, n_2, ..., n_k)$	$P(y)\cdot y^{n_1+n_2+...+n_k}$
5.	$f(z; n, 0, \ldots, 0) = \int_z^1 \frac{\ln^{(k-1)}(x/z)}{(k-1)!}\; x^{n-1} f(x)\, dx$	$P(y)\cdot y^n$
6.	$f(z; n, 1, \ldots, 1) = \int_z^1 \frac{(x-z)^{k-1}}{(k-1)!}\; x^{n-1} f(x)\, dx$	$P(y)\cdot y^{n+k-1}$
7.	$-\frac{df(x)}{dx}$ $[xf(x)]^{(k=1)}$ O.V. Teryaev (2007, 2009)	$P(y) y^{-1}$
8.	$\left(-\frac{d}{dx}\right)^k [x^n f(x)]$	$P(y) y^{n-k}$

Generalization for structure functions

For SF $F = C * f$ one obtains the generalized SF $\mathcal{F}$ and the generalized coefficient function $\mathcal{C}$:

$$F,\ C \to \mathcal{F} = \mathcal{C} * f(z; \{n\}_k), \quad \mathcal{C} = C(y) \cdot y^{n_1+n_2+\ldots+n_k}.$$

The evolution equation for F:

$$\dot{F}(z; \mu^2) = (K * F)(z);$$

$$K = P + \beta(a_s)\,(\partial_{a_s} C) * C^{-1},$$

where β is the QCD β-function, $\mu^2 \frac{d}{d\mu^2} a_s(\mu^2) = \beta(a_s)$.

The generalized evolution equation for $\mathcal{F}$:

$$\dot{\mathcal{F}}(z; \{n\}_k) = \mathcal{K} * \mathcal{F}(z; \{n\}_k), \quad \mathcal{K}(y) = K(y) \cdot y^{n_1+n_2+\ldots+n_k}.$$

Future applications

Based on gCMMA different interesting partial solutions of DGLAP can be constructed and applied to analysis of the experimental data in different restricted x-regions, respectively.

$$f(x_0; \underbrace{n, 0, \ldots, 0}_{k}) = \int_{x_0}^{1} \frac{\ln^{(k-1)}(x/x_0)}{\Gamma(k)}\, x^{n-1} f(x)\, dx$$

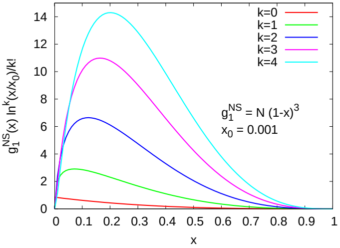
evolves with the kernel which is independent on k:

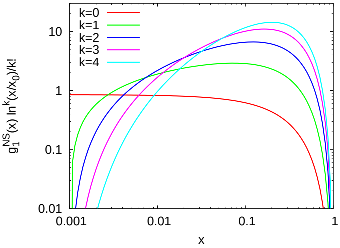
$$\mathcal{P}(y) = P(y) \cdot y^n$$

Integrands $\ln^{(k-1)}(x/x_0)/\Gamma(k)$ at different k are “bricks” for any new gCMM constructions that evolve following the same DGLAP equation.

Future applications BSR

The contribution to $f(x_0; n, 0, \ldots, 0)$ is reinforced at the right end $x = 1$ by powers of logs. This reinforcement becomes especially useful for the case when the experimental data are better known at larger x and, in contrast, ones are unreliable or worse known at lower x.



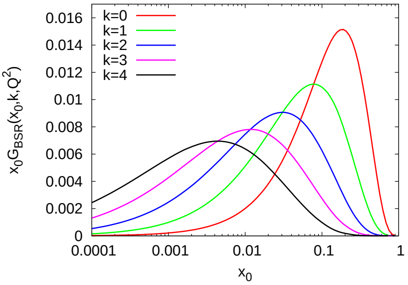


Contributions to the generalized Bjorken Sum Rule

$$G_{BSR}(x_0, k, Q^2) \equiv \int_{x_0}^{1} \frac{\ln^k (x/x_0)}{\Gamma(k+1)}\, g_1^{NS}(x, Q^2)\, dx$$

Future applications BSR

$$G_{BSR}(x_0, k, Q^2) \equiv \int_{x_0}^{1} \frac{\ln^k(x/x_0)}{\Gamma(k+1)}\, g_1^{NS}(x, Q^2)\, dx$$

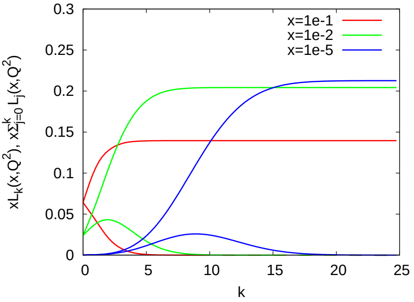


Generalized Bjorken sum rule $x_0 G_{BSR}$ v. x_0 for different k. We assume Regge parametrization at $Q_0^2 = 1\mathrm{GeV}^2$ and LO evolution $Q^2 = 3\,\mathrm{GeV}^2$

Future applications BSR

$$L_k(x,Q^2) \equiv \int_x^1 \frac{\ln^k (z/x)}{k!}\, g_1^{NS}(z,Q^2)\, \frac{dz}{z}$$

$$\int_x^1 g_1^{NS}(z,Q^2)\, dz = x \sum_{k=0}^{\infty} L_k(x,Q^2)$$



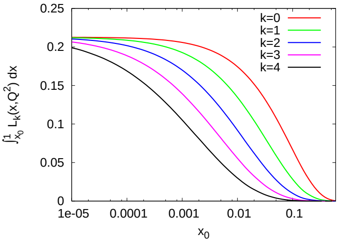
Lower plots: generalized CMM xL_k v. k for different cut points x.
Upper plots: partial sums $x\sum_{j=0}^{k} L_j$

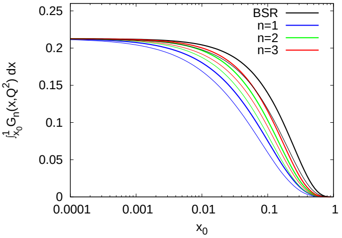
Future applications BSR

$$L_k(x,Q^2) \equiv \int_x^1 \frac{\ln^k(z/x)}{k!}\, g_1^{NS}(z,Q^2)\,\frac{dz}{z}$$

$$G_n(x,Q^2) \equiv n\int_x^1 \left(\frac{x}{z}\right)^{n-1} g_1^{NS}(z,Q^2)\,\frac{dz}{z}$$

$$\int_0^1 L_k(x,Q^2)\,dx = \int_0^1 G_n(x,Q^2)\,dx = \int_0^1 g_1^{NS}(x,Q^2)\,dx = BSR$$





L_k and G_n for each k, n have the same evol. DGLAP kernel $P(y)$ as g_1^{NS}

Recent applications (2008 - 2014)

- determination of the PDFs from CMM
  A.Kotlorz and D.K., Acta Phys. Pol. B40 (2009) 1661
- contributions to BSR and comparison to experimental data HERMES, COMPASS, JLAB
  A.Kotlorz and D.K., Acta Phys. Pol. B39 (2008) 1913
  Phys. Part. Nucl. Lett. 11 (2014) 357
- generalization of the Wandzura-Wilczek relation in terms of CMM
- evolution equation for the structure function g_2
  A.Kotlorz and D.K., Acta Phys. Pol. B42 (2011) 1231
- HT resummation within CMMA (in collab. with O. Teryaev)
  in progress

Conclusions

Generalization of DGLAP equations is obtained

General CMM (multiple integrations as well as multiple differentiations of the original parton distribution) obey the same DGLAP evol. eqs. with simply modified evolution kernel

Advantages of CMM

Fundamental properties of the nucleon can be studied in a restricted experimentally range of Bjorken-x

EXPERIMENTS PROVIDE CUT MOMENTS

No uncertainties from the unmeasurable regions!

Novel tools providing a rich variety of further possible ways to test QCD

Choice of the suitable classes of CMM for the available experimental kinematic range

Enhancement of x-region with smaller uncertainties

Спасибо Дубна!