QCD inspired meson model and Swinger-Dyson equation for massless quark.

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Abstract

We present ideas that are usually not taken into account in QCD studies: importance of formulation in Minkowski-spacetime and effect of an operator product expansion by means of normal ordering of fields in lagrangian. We demonstrate a possible way from QCD lagrangian to effective action of strong interaction. Then we derive a Schwinger–Dyson equation for quark and study it both analytically and numerically. We consider a simplest possible model, but methods and ideas can be used in more general case.

Outline:

Effective Action of Strong Interaction

We start with $N_c = 3$, $N_f = 1$:

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - A^a_\mu j^{a\mu} + \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \qquad j^{a\mu} = -g\overline{\psi}\gamma^\mu\frac{\lambda^a}{2}\psi$$

We want to derive from this lagrangian an effective action for meson-like bound state[1], under some *restrictions* and *assumptions*:

- A first restriction is a choice of frame of reference. Below after some calculation we obtain a bound state which at whole will be at rest in this frame of reference. So only *static* problems considered.

Gauge: $\partial_k A^a_k(x) = 0$

[1] There were a lot of attempts of making this: Arbuzov, Volkov, Efimov, Ivanov, Nedelko, etc.

After quantization the A_μ^a become an operator field. Vacuum 2-point correlator:

$$\langle 0|A_i^a(x)A_j^b(x)|0\rangle = 2C_g\delta_{ij}\delta^{ab}$$

- Suppose that $C_g \neq 0$ and $C_g < \infty$. In fact C_g should depend on energy, but we also suppose that C_g is constant.

Assuming $C_g \neq 0$, make normal ordering in lagrangian (standard approach is $C_g = 0$). The gluon term[2]:

$$-\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} = \frac{1}{2}\dot{A}_i^a\dot{A}_i^a - \frac{1}{4}F_{ij}^a F^{aij} + \frac{1}{2}A_0^a(-\Delta + M_g^2)A_0^a + \dots$$

where: $M_g^2 \equiv 6g^2C_gN_c$.

- Let us consider dotted terms as perturbation.

[2] That is consistent with phenomenology that at small energies gluon effectively have mass (Scadron, Politzer, Zakharov, etc.).

Generating functional:

$$\mathcal{Z} = \int \mathsf{D}A_\mu^a \delta(\partial_k A_k^a) \mathsf{D}\overline{\psi} \mathsf{D}\psi \exp\Big[i \int d^4x \Big(\frac{1}{2} \dot{A}_i^a \dot{A}_i^a - \frac{1}{4} F_{ij}^a F^{aij} + \\ + \frac{1}{2} A_0^a (-\Delta + M_g^2) A_0^a - A_0^a j_0^a + A_i^a j_i^a + \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi \Big) + \\ + i \int d^4x (A_i^a J^{ai} + \overline{\eta}\psi + \overline{\psi}\eta) \Big]$$

Make integration over A_0^a.

A Fierz-like transform of the color Gell-Mann matrices:

$$\frac{\lambda^{a r_1 r_2}}{2} \frac{\lambda^{a s_2 s_1}}{2} = \frac{1}{3} \delta^{r_1 s_1} \delta^{r_2 s_2} + \frac{1}{6} \varepsilon^{t r_1 s_2} \varepsilon^{t s_1 r_2}$$

- As we want to consider only colorless mesons, we neglect the second term.

Rewrite using $\mathcal{K}$: $-\frac{1}{2}\int d^4x\, d^4y\; j_0^a(x)\delta(x^0-y^0)\frac{1}{4\pi}\frac{e^{-M_g|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\; j_0^a(y) =$

$$= -\frac{1}{2}\int d^4x_1\, d^4x_2\, d^3\mathbf{y}_1\, d^3\mathbf{y}_2\; \overline{\psi}^{r_1}_{\alpha_1}(x_1)\,\psi^{\alpha_2 r_2}(x_2)\delta^{r_1 s_1}\times$$

$$\underbrace{\times\gamma^{0\,\alpha_1}{}_{\alpha_2}\,\delta^4(x_1-x_2)\,\frac{g^2}{12\pi}\,\frac{e^{-M_g|\mathbf{x}_1-\mathbf{y}_2|}}{|\mathbf{x}_1-\mathbf{y}_2|}\,\delta^3(\mathbf{y}_1-\mathbf{y}_2)\,\gamma^{0\,\beta_2}{}_{\beta_1}\times}_{\mathcal{K}^{\alpha_1}{}_{\beta_1\alpha_2}{}^{\beta_2}(x_1,\mathbf{y}_1;x_2,\mathbf{y}_2)}$$

$$\times\delta^{r_2 s_2}\,\overline{\psi}^{s_2}_{\beta_2}(x_2^0,\mathbf{y}_2)\,\psi^{\beta_1 s_1}(x_1^0,\mathbf{y}_1)+\ldots$$

- Let's consider $\psi^{\alpha s}(x^0,\mathbf{x})\overline{\psi}^s_\beta(x^0,\mathbf{y})$ as a real bilocal field.

The color index s is summing inside the pair $\psi\overline{\psi}$, so the pair $\psi\overline{\psi}$ as whole is colorless.

Introduce new bilocal field $\mathcal{M}^\alpha{}_\beta(x^0,\mathbf{x},\mathbf{y})$ and make a bosonization (Habbard-Stratanovich) transform.

Finally:

$$\mathcal{Z} = \int \mathsf{D} A_k^a \delta(\partial_k A_k^a) \mathsf{D}\overline{\psi} \mathsf{D}\psi \mathsf{D}\mathcal{M}$$

$$\exp\Big[i \int d^4x \Big(\frac{1}{2} \dot{A}_i^a \dot{A}_i^a - \frac{1}{4} F_{ij}^a F^{aij} + A_i^a j_i^a + \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi \Big) +$$

$$+ \frac{i}{2} \int d^4x_1\, d^4x_2\, d^3\mathbf{y}_1\, d^3\mathbf{y}_2\, {\mathcal{M}^{\mathrm{T}}}_{\alpha_1}{}^{\beta_1}(x_1^0, \mathbf{x}_1, \mathbf{y}_1) \times$$

$$\times\, {\mathcal{K}^{-1}}^{\alpha_1}{}_{\beta_1 \alpha_2}{}^{\beta_2}(x_1, \mathbf{y}_1; x_2, \mathbf{y}_2)\, \mathcal{M}^{\alpha_2}{}_{\beta_2}(x_2^0, \mathbf{x}_2, \mathbf{y}_2) +$$

$$+ i \int d^4x\, d^3\mathbf{y}\, \overline{\psi}_\alpha(x^0, \mathbf{x})\, \psi^\beta(x^0, \mathbf{y})\, \mathcal{M}^\alpha{}_\beta(x^0, \mathbf{x}, \mathbf{y}) +$$

$$+ i \int d^4x (A_i^a J^{ai} + \overline{\eta}\psi + \overline{\psi}\eta) \Big]$$

For quantization of Bilocal Fields we use Stationary Phase method (semiclassical approximation).

After integrating over fermions:

$$\mathcal{Z} = \int \mathsf{D} A_k^a \delta(\partial_k A_k^a) \mathsf{D}\mathcal{M} e^{iS_{eff}}$$

Swinger-Dyson (Gap) equation – fermion spectrum[3]:

$$\frac{\delta S_{eff}}{\delta \mathcal{M}}(A_k^a = 0, \overline{\eta} = 0, \eta = 0, J = 0) = 0$$

Search the solution in the form

$$\mathcal{M}^{\alpha}{}_{\beta}(x^0, \mathbf{x}, \mathbf{y}) \;=\; -\Sigma^{\alpha}{}_{\beta}(x^0, \mathbf{x}, \mathbf{y}) \;+\; m\delta^{\alpha}{}_{\beta}\, \delta^3(\mathbf{x}-\mathbf{y})$$

[3] Bethe-Salpeter equation – bound state spectrum:

$$\frac{\delta^2 S_{eff}}{\delta \mathcal{M}^2}(\mathcal{M} = -\Sigma + m)\Gamma \;=\; 0$$

The Swinger-Dyson equation takes form:

$$\Sigma^{\alpha_1}{}_{\beta_1}(x_1^0,\mathbf{x}_1,\mathbf{y}_1) \;=\; m\,\delta^{\alpha_1}{}_{\beta_1}\,\delta^3(\mathbf{x}_1-\mathbf{y}_1) + \\ + 3i\int d^4x_2\,d^4y_2\,\mathcal{K}^{\alpha_1}{}_{\beta_1\alpha_2}{}^{\beta_2}(x_1,\mathbf{y}_1;x_2,\mathbf{y}_2)\,G_\Sigma{}^{\alpha_2}{}_{\beta_2}(x_2,y_2)\,\delta(x_2^0-y_2^0)$$

where: $G_\Sigma^{-1\,\alpha}{}_\beta(x,y) = i\gamma^{\mu\,\alpha}{}_\beta\partial_\mu\,\delta^4(x-y) \;-\; \Sigma^\alpha{}_\beta(x^0,\mathbf{x},\mathbf{y})\,\delta(x^0-y^0)$

- Try next ansatz:

$$\Sigma^\alpha{}_\beta(x^0,\mathbf{x},\mathbf{y}) = \delta^\alpha{}_\beta\,\frac{1}{(2\pi)^{\frac{3}{2}}}M(\mathbf{x}-\mathbf{y})$$

After Fourier-transform:

$$M(\mathbf{p})\,\delta^{\alpha_1}{}_{\beta_1} \;=\; m\,\delta^{\alpha_1}{}_{\beta_1} \;- \\ -\,i\frac{g^2}{(2\pi)^4}\int d^4q\;\frac{1}{(\mathbf{p}-\mathbf{q})^2+M_g^2}\;\gamma^{0\alpha_1}{}_{\alpha_2}\,G_\Sigma{}^{\alpha_2}{}_{\beta_2}(q)\,\gamma^{0\beta_2}{}_{\beta_1}$$

where:

$$G_\Sigma(q) = e^{-\gamma^i\frac{q_i}{|\mathbf{q}|}\varphi(\mathbf{q})}\left(\frac{1}{q_0+E(\mathbf{q})-i\varepsilon}\cdot\frac{1+\gamma^0}{2}\;+ \\ +\;\frac{1}{q_0-E(\mathbf{q})+i\varepsilon}\cdot\frac{1-\gamma^0}{2}\right)e^{\gamma^i\frac{q_i}{|\mathbf{q}|}\varphi(\mathbf{q})}\gamma^0$$

$$E(\mathbf{q}) \equiv \sqrt{M(\mathbf{q})^2+\mathbf{q}^2} \qquad\qquad \cos 2\varphi(\mathbf{q}) \equiv \frac{M(\mathbf{q})}{E(\mathbf{q})}$$

We can see that one can direct integrate over q_0.
After integrating over solid angles, finally:

$$\boxed{M(p) = m + \frac{g^2}{(4\pi)^2}\frac{1}{p}\int\limits_0^\infty dq\frac{qM(q)}{\sqrt{M^2(q)+q^2}}\ln\left(\frac{M_g^2+(p+q)^2}{M_g^2+(p-q)^2}\right)}$$

Solution of the massless Schwinger-Dyson equation

- $m = 0$.
- $M(q) \to 0$ at $q \to \infty$, ($\Rightarrow$ no renormalization is need).

Introduce dimensionless variables:

$$\bar{p} \equiv \frac{p}{M_g} \quad \bar{q} \equiv \frac{q}{M_g} \quad \bar{M}(\bar{p}) \equiv \frac{M(p)}{M_g}$$

The Schwinger-Dyson equation takes form:

$$\bar{M}(\bar{p}) = \frac{g^2}{(4\pi)^2}\frac{1}{\bar{p}}\int\limits_0^\infty d\bar{q}\frac{\bar{q}\bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q})+\bar{q}^2}}\ln\left(\frac{1+(\bar{p}+\bar{q})^2}{1+(\bar{p}-\bar{q})^2}\right)$$

There is always a solution: $\bar{M}(\bar{p}) = 0$

Put by definition $\bar{M}(-\bar{p}) = \bar{M}(\bar{p})$, than:

$$\bar{M}(\bar{p}) = \frac{g^2}{2(4\pi)^2}\frac{1}{\bar{p}}\int\limits_{-\infty}^{+\infty} d\bar{q}\frac{\bar{q}\bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q})+\bar{q}^2}}\ln\left(\frac{1+(\bar{p}+\bar{q})^2}{1+(\bar{p}-\bar{q})^2}\right)$$

Numerical solution

Things that should be avoided:

1. Upper limit of integration must be $+\infty$, and can not be replaced by finite quantity Λ.
2. $\bar{M}(+\infty) = 0$, otherwise integral diverge.
3. It is better to avoid replacing continuous function $\bar{M}(\bar{p})$ by a discrete table $\bar{M}(\bar{p}_i)$ with fixed numbers of points $\bar{p}_i$.

Introduce:

$$W(\bar{q}) \equiv \frac{\bar{q}\bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2}}$$

Introduce new variables: $\bar{p} = \lambda \tan\frac{\varphi}{2} \quad , \quad \varphi \in (-\pi, \pi)$

$$\bar{q} = \lambda \tan\frac{\theta}{2} \quad , \quad \theta \in (-\pi, \pi)$$

where λ - some parameter.

Schwinger-Dyson equation takes form:

$$\bar{M}(\varphi) = \frac{g^2}{2(4\pi)^2} \int\limits_{-\pi}^{+\pi} \frac{d\theta}{2\,\tan\frac{\varphi}{2}\,\cos^2\frac{\theta}{2}} \ln\left(\frac{1+\lambda^2(\tan\frac{\varphi}{2}+\tan\frac{\theta}{2})^2}{1+\lambda^2(\tan\frac{\varphi}{2}-\tan\frac{\theta}{2})^2}\right) W(\theta)$$

On $[-\pi,\pi]$ there is convenient system of functions – Fourier series:

$$\bar{M}(\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cdot \cos k\varphi \qquad W(\theta) = \sum_{k=1}^{\infty} b_k \cdot \sin k\theta$$

The equation:

$$a_k = A_{kj} b_j$$

where: $A_{kj} \equiv \frac{g^2}{32\pi^3} M_{kj}$, where:

$$M_{kj} \equiv \int\limits_{-\pi}^{+\pi} d\varphi \int\limits_{-\pi}^{+\pi} d\theta \frac{\cos(k\varphi)}{2\,\tan\frac{\varphi}{2}\,\cos^2\frac{\theta}{2}} \ln\left(1+\frac{\lambda^2 \sin\varphi\,\sin\theta}{(\cos\frac{\varphi}{2}\,\cos\frac{\theta}{2})^2+(\lambda\sin\frac{\varphi-\theta}{2})^2}\right)\sin(j\theta)$$

$W_0(\varphi)$

$\downarrow$

to Fourier coeff. b_j

$\downarrow$

$a_k = A_{kj} b_j \quad \longrightarrow \quad \bar{M}(\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cdot \cos k\varphi$

$\uparrow \qquad\qquad\qquad\qquad \downarrow \qquad\qquad \searrow$

to Fourier coeff. $b_j \quad \longleftarrow \quad W(\varphi) = \frac{\lambda \sin \frac{\varphi}{2} \bar{M}(\varphi)}{\sqrt{(\cos \frac{\varphi}{2} \bar{M}(\varphi))^2 + (\lambda \sin \frac{\varphi}{2})^2}}$ Result

$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \nearrow$

$W(\theta) = \sum_{k=1}^{\infty} b_k \cdot \sin k\theta \quad \longrightarrow$ Insert in exact SD equation

There is only $\bar{M}(\bar{p}) = 0$ solution at $g^2 < 16$.

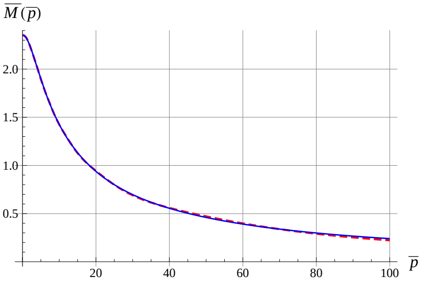

Figure: Plot of $\bar{M}(\bar{p})$, at $g^2 = 18$, $\lambda = 10$, 13 harmonics.

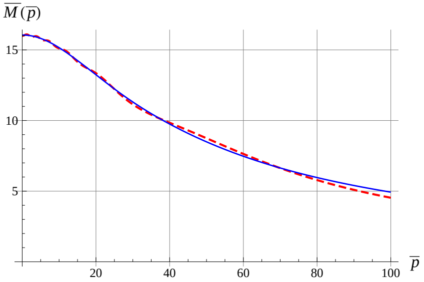

Figure: Plot of $\bar{M}(\bar{p})$, at $g^2 = 19$, $\lambda = 10$, 13 harmonics.

Analytical estimations.

Try to find asymptotical behavior of $\bar{M}(\bar{p})$ at large $\bar{p}$.

The Schwinger-Dyson equation:

$$\bar{M}(\bar{p}) = \frac{g^2}{2(4\pi)^2}\frac{1}{\bar{p}}\int\limits_{-\infty}^{+\infty} d\bar{q}\frac{\bar{q}\bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q})+\bar{q}^2}}\ln\left(\frac{1+(\bar{p}+\bar{q})^2}{1+(\bar{p}-\bar{q})^2}\right)$$

$$\Big\Updownarrow{\scriptstyle \bar{q}\mapsto -\bar{q}}$$

$$\bar{M}(\bar{p}) = -\frac{g^2}{(4\pi)^2}\frac{1}{\bar{p}}\int\limits_{-\infty}^{+\infty} d\bar{q}\frac{\bar{q}\bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q})+\bar{q}^2}}\ln\Big(1+(\bar{p}-\bar{q})^2\Big)$$

We try to solve approximate equation, where $\bar{M}_0 \equiv \bar{M}(0)$:

$$\bar{M}(\bar{p}) = -\frac{g^2}{(4\pi)^2}\frac{1}{\bar{p}}\int\limits_{-\infty}^{+\infty} d\bar{q}\frac{\bar{q}\bar{M}(\bar{q})}{\sqrt{\bar{M}_0^2+\bar{q}^2}}\ln\Big(1+(\bar{p}-\bar{q})^2\Big)$$

Introduce: $\mathcal{W}(\bar{q}) \equiv \dfrac{\bar{q}\bar{M}(\bar{q})}{\sqrt{\bar{M}_0^2 + \bar{q}^2}}$

$$\sqrt{\bar{M}_0^2 + \bar{p}^2}\,\mathcal{W}(\bar{p}) \;=\; -\frac{g^2}{16\pi^2}\int\limits_{-\infty}^{+\infty} d\bar{q}\;\ln\Big(1 + (\bar{p} - \bar{q})^2\Big)\,\mathcal{W}(\bar{q})$$

After Fourier transform:

$$\sqrt{\bar{M}_0^2 - \partial^2}\,\mathcal{W}(x) \;=\; \frac{g^2}{8\pi}\,\frac{e^{-|x|}}{|x|}\,\mathcal{W}(x)$$

Consider $\bar{p} \to \infty$ asymptotics.

$$\sqrt{\bar{M}_0^2 + \bar{p}^2} \;\longrightarrow\; |\bar{p}|$$

It corresponds $x \to 0$, so Taylor expansion can be used.

Try next ansatz:

$$\bar{M}(\bar{p}) = C\frac{1}{|\bar{p}|^{\beta}}$$

The SD equation is self-consistent if:

$$\boxed{\frac{1}{g^2} = \frac{1}{8\pi}\frac{\cot\left(\frac{\pi\beta}{2}\right)}{(1-\beta)}}$$

Hence: $g^2 \leqslant 16$

$0 < \beta < 2$

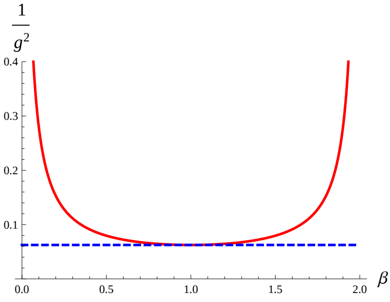

Figure: Plot $\frac{1}{8\pi}\frac{\cot(\frac{\pi\beta}{2})}{(1-\beta)}$ and $\frac{1}{16}$.

Let us subtract from equation the asymptotic:

$$\bar{M}(\bar{p}) - C\frac{1}{\bar{p}^{\beta}} = \frac{g^2}{(4\pi)^2}\frac{1}{\bar{p}}\int_0^{\infty} d\bar{q}\frac{\bar{q}\bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q})+\bar{q}^2}}\ln\left(\frac{1+(\bar{p}+\bar{q})^2}{1+(\bar{p}-\bar{q})^2}\right) - C\int_0^{\infty} d\bar{q}\frac{\beta}{(\bar{p}+\bar{q})^{\beta}}$$

After integration over $\bar{p}$ and substitution $g^2 < 16$:

$$\int_0^{\infty} d\bar{p}\left(\bar{M}(\bar{p}) - C\frac{1}{\bar{p}^{\beta}}\right) < \int_0^{\infty} d\bar{q}\left(\frac{\bar{q}\bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q})+\bar{q}^2}} - C\frac{1}{\bar{q}^{\beta}}\right)$$

what is a impossible.
Therefore there is no asymptotic with $0 < \beta < 1$.

Analogously one can prove that asymptotic with $0 < \beta < 1$ also is not exist.

The nontrivial solution exist only at $g^2 = 16$.[4]
At $\bar{p} \to \infty$:

$$\bar{M}(\bar{p}) \sim \frac{1}{\bar{p}}$$

[4] $\alpha_s = \frac{g^2}{4\pi} = \frac{4}{\pi}$

Conclusions

1. The simple model of strong interaction with massive gluon was constructed.
2. In framework of this model we studied the Swinger-Dyson equation for quark analytically and numerically and got that nontrivial solution appear only at $g^2 = 16$ with asymptotic $\bar{M}(\bar{p}) \sim \frac{1}{\bar{p}}$.
3. The created programs for numerical calculation of the Swinger-Dyson equation can be used not only in our case but for various kernels.