Hadronization and Solvable Models of Renormdynamics of QCD

Nugzar Makhaldiani

JINR Dubna, mnv@jinr.ru

Talk presented at the XXII International Baldin Seminar on High Energy Physics Problems
"Relativistic Nuclear Physics and Quantum Chromodynamics"
September 15 to 20, 2014 in Dubna, Russia.

Quantum field theory (QFT) and Fractal calculus (FC) provide Universal language of fundamental physics (see e.g. [Makhaldiani, 2011]). In QFT existence of a given theory means, that we can control its behavior at some scales (short or large distances) by renormalization theory [Collins, 1984]. If the theory exists, than we want to solve it, which means to determine what happens on other (large or short) scales. This is the problem (and content) of Renormdynamics. The result of the Renormdynamics, the solution of its discrete or continual motion equations, is the effective QFT on a given scale (different from the initial one).

Perturbation theory series (PTS) have the following qualitative form

$$f(g) = f_0 + f_1 g + ... + f_n g^n + ..., \; f_n = n!P(n)$$
$$f(x) = \sum_{n\geq 0} P(n)n!x^n = P(\delta)\Gamma(1+\delta)\frac{1}{1-x}, \; \delta = x\frac{d}{dx} \qquad (1)$$

So, we reduce previous series to the standard geometric progression series. This series is convergent for $|x| < 1$ or for $|x|_p = p^{-k} < 1, \; x = p^k a/b, \; k \geq 1$. With proper nomalization of the expansion parametre, the coefficients of the series are rational numbers and if experimental data indicates for some prime value for g, e.g. in QED

$$g = \frac{e^2}{4\pi} = \frac{1}{137.0...} \qquad (2)$$

then we can take corresponding prime number and consider p-adic convergence of the series. In the case of QED, we have

$$f(g) = \sum f_n p^{-n}, \; f_n = n!P(n), \; p = 137, \; |f|_p \leq \sum |f_n|_p p^n \qquad (3)$$

In the Yukawa theory of strong interactions (see e.g. [Bogoliubov, Shirkov, 1959]), we take $g = 13$,

$$f(g) = \sum f_n p^n, \ f_n = n!P(n), \ p = 13,$$
$$|f|_p \leq \sum |f_n|_p p^{-n} < \frac{1}{1 - p^{-1}} \tag{4}$$

So, the series is convergent. If the limit is rational number, we consider it as an observable value of the corresponding physical quantity.
In $MSSM$ (see [Kazakov, 2004]) coupling constants unifies at $\alpha_u^{-1} = 26.3 \pm 1.9 \pm 1$. So,

$$23.4 < \alpha_u^{-1} < 29.2 \tag{5}$$

Question: how many primes are in this interval?

$$24, \ 25, \ 26, \ 27, \ 28, \ 29 \tag{6}$$

Only one!
Proposal: take the value $\alpha_u^{-1} = 29.0...$ which will be two orders of magnitude more precise prediction and find the consequences for the SM scale observables.

Let us make more explicit the formal representation of (1)

$$f(x) = \sum_{n\geq 0} P(n)n!x^n = P(\delta)\Gamma(1+\delta)\frac{1}{1-x},$$
$$= P(\delta)\int_0^\infty dte^{-t}t^\delta \frac{1}{1-x} = P(\delta)\int_0^\infty dt\frac{e^{-t}}{1+(-x)t},\ \delta = x\frac{d}{dx} \quad (7)$$

This integral is well defined for negative values of x. The Mathematica answer for the corresponding integral is

$$I(x) = \int_0^\infty dt\frac{e^{-t}}{1+xt} = e^{1/x}\Gamma(0,1/x)/x,\ Im(x)\neq 0,\ Re(x)\geq 0, \quad (8)$$

where $\Gamma(a,z)$ is the incomplete gamma function

$$\Gamma(a,z) = \int_z^\infty dtt^{a-1}e^{-t} \quad (9)$$

For $x = 0.001,\ I(x) = 0.999$

The Goldberger-Treiman relation (GTR) [Goldberger, Treiman, 1958] plays an important role in theoretical hadronic and nuclear physics. GTR relates the Meson-Nucleon coupling constants to the axial-vector coupling constant in β-decay:

$$g_{\pi N} f_\pi = g_A m_N \tag{10}$$

where m_N is the nucleon mass, g_A is the axial-vector coupling constant in nucleon β-decay at vanishing momentum transfer, f_π is the π decay constant and $g_{\pi N}$ is the $\pi - N$ coupling constant.
Since the days when the Goldberger-Treiman relation was discovered, the value of g_A has increased considerably. Also, f_π decreased a little, on account of radiative corrections. The main source of uncertainty is $g_{\pi N}$.

If we take

$$\alpha_{\pi N} = \frac{g_{\pi N}^2}{4\pi} = 13 \Rightarrow g_{\pi N} = 12.78 \quad (11)$$

the proton mass $m_p = 938MeV$ and $f_\pi = 93MeV$, from (10), we find

$$g_A = \frac{f_\pi g_{\pi N}}{m_N} = \frac{93 \times \sqrt{52\pi}}{938} = 1.2672 \quad (12)$$

which is in agreement with contemporary experimental value
$g_A = 1.2695(29)$
In an old version of the unified theory [Heisenberg 1966], for the $\alpha_{\pi N}$ the following value were found

$$\alpha_{\pi N} = 4\pi(1 - \frac{m_\pi^2}{3m_p^2}) = 12.5 \quad (13)$$

Determination of $g_{\pi N}$ from $NN, N\bar{N}$ and πN data by the Nijmegen group [Rentmeester et al, 1999] gave the following value

$$g_{\pi N} = 13.05 \pm .08,\ \Delta = 1 - \frac{g_A m_N}{g_{\pi N} f_\pi} = .014 \pm .009,$$
$$13.39 < \alpha_{\pi N} < 13.72 \quad (14)$$

This value is consistent with assumption $g_{\pi N} = 13 \Rightarrow \alpha_{\pi N} = 13.45$
Due to the smallness of the u and d quark masses, Δ is necessarily very small, and its determination requires a very precise knowledge of the $g_{\pi N}$ coupling (g_A and f_π are already known to enough precision, leaving most of the uncertainty in the determination of Δ to the uncertainty in $g_{\pi N}$).

QCD is the theory of the strong interactions with, as only inputs, one mass parameter for each quark species and the value of the QCD coupling constant at some energy or momentum scale in some renormalization scheme. This last free parameter of the theory can be fixed by Λ_{QCD}, the energy scale used as the typical boundary condition for the integration of the Renormdynamic (RD) equation for the strong coupling constant. This is the parameter which expresses the scale of strong interactions, the only parameter in the limit of massless quarks. While the evolution of the coupling with the momentum scale is determined by the quantum corrections induced by the renormalization of the bare coupling and can be computed in perturbation theory, the strength itself of the interaction, given at any scale by the value of the renormalized coupling at this scale, or equivalently by Λ_{QCD}, is one of the above mentioned parameters of the theory and has to be taken from experiment.

The RD equations play an important role in our understanding of Quantum Chromodynamics and the strong interactions. The beta function and the quarks mass anomalous dimension are among the most prominent objects for QCD RD equations. The calculation of the one-loop β-function in QCD has lead to the discovery of asymptotic freedom in this model and to the establishment of QCD as the theory of strong interactions
['t Hooft, 1972, Gross, Wilczek, 1973, Politzer, 1973].
The MS-scheme ['t Hooft, 1973] belongs to the class of massless schemes where the β-function does not depend on masses of the theory and the first two coefficients of the β-function are scheme-independent.

The Lagrangian of QCD with massive quarks in the covariant gauge is

$$L = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{q}_n(i\gamma D - m_n)q_n$$
$$-\frac{1}{2\xi}(\partial A)^2 + \partial^\mu \bar{c}^a(\partial_\mu c^a + gf^{abc}A^b_\mu c^c)$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu,\ (D_\mu)_{kl} = \delta_{kl}\partial_\mu - igt^a_{kl}A^a_\mu, \quad (15)$$

$A^a_\mu, a = 1, ..., N_c^2 - 1$ are gluon; $q_n, n = 1, ..., n_f$ are quark; c^a are ghost fields; ξ is gauge parameter; t^a are generators of fundamental representation and f^{abc} are structure constants of the Lie algebra $[t^a, t^b] = if^{abc}t^c$, we consider an arbitrary compact semi-simple Lie group G. For QCD, $G = SU(N_c), N_c = 3$.
The RD equation for the coupling constant is

$$\dot{a} = \beta(a) = \beta_2 a^2 + \beta_3 a^3 + \beta_4 a^4 + \beta_5 a^5 + O(a^6),$$
$$a = \frac{\alpha_s}{4\pi} = (\frac{g}{4\pi})^2,\ \int_{a_0}^a \frac{da}{\beta(a)} = t - t_0 = \ln\frac{\mu^2}{\mu_0^2}, \quad (16)$$

μ is the 't Hooft unit of mass, the renormalization point in the MS-scheme.

To calculate the β-function we need to calculate the renormalization constant Z of the coupling constant, $a_b = Za$, where a_b is the bare (unrenormalized) charge. The expression of the β-function can be obtained in the following way

$$0 = d(a_b\mu^{2\varepsilon})/dt = \mu^{2\varepsilon}(\varepsilon Za + \frac{\partial(Za)}{\partial a}\frac{da}{dt})$$
$$\Rightarrow \frac{da}{dt} = \beta(a,\varepsilon) = \frac{-\varepsilon Za}{\frac{\partial(Za)}{\partial a}} = -\varepsilon a + \beta(a),\ \beta(a) = a\frac{d}{da}(aZ_1) \quad (17)$$

where

$$\beta(a,\varepsilon) = \frac{D-4}{2}a + \beta(a) \quad (18)$$

is $D-$dimensional $\beta-$function and Z_1 is the residue of the first pole in ε expansion

$$Z(a,\varepsilon) = 1 + Z_1\varepsilon^{-1} + ... + Z_n\varepsilon^{-n} + ... \quad (19)$$

Since Z does not depend explicitly on μ, the β-function is the same in all MS-like schemes, i.e. within the class of renormalization schemes which differ by the shift of the parameter μ.

For quark anomalous dimension, RD equation is

$$\dot{b} = \gamma(a) = \gamma_1 a + \gamma_2 a^2 + \gamma_3 a^3 + \gamma_4 a^4 + O(a^5),$$
$$b(t) = b_0 + \int_{t_0}^{t} dt\gamma(a(t)) = b_0 + \int_{a_0}^{a} da\gamma(a)/\beta(a). \quad (20)$$

To calculate the quark mass anomalous dimension $\gamma(g)$ we need to calculate the renormalization constant Z_m of the quark mass $m_b = Z_m m,\ m_b$ is the bare (unrenormalized) quark mass. Than we find the function $\gamma(g)$ in the following way

$$0 = \dot{m}_b = \dot{Z}_m m + Z_m \dot{m} = Z_m m((\ln Z_m)^{\cdot} + (\ln m)^{\cdot})$$
$$\Rightarrow \gamma(a) = -\frac{d\ln Z_m}{dt} = \dot{b} = -\frac{d\ln Z_m}{da}\frac{da}{dt} = -\frac{d\ln Z_m}{da}(-\varepsilon a + \beta(a))$$
$$= a\frac{dZ_{m1}}{da},\ b = -\ln Z_m = \ln\frac{m}{m_b}, \quad (21)$$

where RD equation in $D-$dimension is

$$\dot{a} = -\varepsilon a + \beta(a) = \beta_1 a + \beta_2 a^2 + ... \tag{22}$$

and Z_{m1} is the coefficient of the first pole in the $\varepsilon-$expantion of the Z_m in MS-scheme

$$Z_m(\varepsilon, g) = 1 + Z_{m1}(g)\varepsilon^{-1} + Z_{m2}(g)\varepsilon^{-2} + ... \tag{23}$$

Since Z_m does not depend explicitly on μ and m, the γ_m-function is the same in all MS-like schemes.

RD equation,

$$\dot{a} = \beta_1 a + \beta_2 a^2 + ... \tag{24}$$

can be reparametrized,

$$a(t) = f(A(t)) = A + f_2 A^2 + ... + f_n A^n + ... = \sum_{n\geq 1} f_n A^n, \tag{25}$$

$$\dot{A} = b_1 A + b_2 A^2 + ... = \sum_{n\geq 1} b_n A^n,$$

$$\begin{aligned}
&\dot{a} = \dot{A} f'(A) = (b_1 A + b_2 A^2 + ...)(1 + 2f_2 A + ... + n f_n A^{n-1} + ...) \\
&= \beta_1 (A + f_2 A^2 + ... + f_n A^n + ...) + \beta_2 (A^2 + 2f_2 A^3 + ...) + ... \\
&+ \beta_n (A^n + n f_2 A^{n+1} + ...) + ... \\
&= \beta_1 A + (\beta_2 + \beta_1 f_2) A^2 + (\beta_3 + 2\beta_2 f_2 + \beta_1 f_3) A^3 + \\
&... + (\beta_n + (n-1)\beta_{n-1} f_2 + ... + \beta_1 f_n) A^n + ... \\
&= \sum_{n,n_1,n_2\geq 1} A^n b_{n_1} n_2 f_{n_2} \delta_{n,n_1+n_2-1}
\end{aligned} \tag{26}$$

$$= \sum_{n,m\geq 1; m_1,...,m_k \geq 0} A^n \beta_m f_1^{m_1} ... f_k^{m_k} f(n,m,m_1,...,m_k),$$

$$f(n,m,m_1,...,m_k) = \frac{m!}{m_1!...m_k!} \delta_{n,m_1+2m_2+...+km_k} \delta_{m,m_1+m_2+...+m_k},$$
$$b_1 = \beta_1,\ b_2 = \beta_2 + f_2\beta_1 - 2f_2b_1 = \beta_2 - f_2\beta_1,$$
$$b_3 = \beta_3 + 2f_2\beta_2 + f_3\beta_1 - 2f_2b_2 - 3f_3b_1 = \beta_3 + 2(f_2^2 - f_3)\beta_1,$$
$$b_4 = \beta_4 + 3f_2\beta_3 + f_2^2\beta_2 + 2f_3\beta_2 - 3f_4b_1 - 3f_3b_2 - 2f_2b_3, ...$$
$$b_n = \beta_n + ... + \beta_1 f_n - 2f_2b_{n-1} - ... - nf_nb_1, ... \quad (27)$$

so, by reparametrization, beyond the critical dimension ($\beta_1 \neq 0$) we can change any coefficient but β_1.

We can fix any higher coefficient with zero value, if we take

$$f_2 = \frac{\beta_2}{\beta_1},\ f_3 = \frac{\beta_3}{2\beta_1} + f_2^2,\ \dots,f_n = \frac{\beta_n + \dots}{(n-1)\beta_1}, \dots \tag{28}$$

In the critical dimension of space-time, $\beta_1 = 0$, and we can change by reparametrization any coefficient but β_2 and β_3.
From the relations (27), in the critical dimension ($\beta_1 = 0$), we find that, we can define the minimal form of the RD equation

$$\dot{A} = \beta_2 A^2 + \beta_3 A^3, \tag{29}$$

We can solve (29) as implicit function,

$$u^{\beta_3/\beta_2} e^{-u} = c e^{\beta_2 t},\ u = \frac{1}{A} + \frac{\beta_3}{\beta_2} \tag{30}$$

then, as in the noncritical case, explicit solution will be given by reparametrization representation (25) [Makhaldiani, 2013].

If we know somehow the coefficients β_n, e.g. for first several exact and for others asymptotic values (see e.g. [Kazakov, Shirkov, 1980]) than we can construct reparametrization function (25) and find the dynamics of the running coupling constant. This is similar to the action-angular canonical transformation of the analytic mechanics (see e.g. [Faddeev, Takhtajan, 1990]).
Statement: The reparametrization series for a is p-adically convergent, when β_n and A are rational numbers.

Let us take the the anomalous dimension of some quantity

$$\gamma(a) = \gamma_1 a + \gamma_2 a^2 + \gamma_3 a^3 + ... \quad (31)$$

and make reparametrization

$$a = f(A) = A + f_2 A^2 + f_3 A^3 + ... \quad (32)$$

$$\gamma(a) = \gamma_1(A + f_2 A^2 + f_3 A^3 + ...) + \gamma_2(A^2 + 2 f_2 A^3 + ...) + \gamma_3(A^3 + ...)$$
$$= \Gamma_1 A + \Gamma_2 A^2 + \Gamma_3 A^3 + ...$$
$$\Gamma_1 = \gamma_1, \ \Gamma_2 = \gamma_2 + \gamma_1 f_2, \ \Gamma_3 = \gamma_3 + 2\gamma_2 f_2 + \gamma_1 f_3, ...$$

When $\gamma_1 \neq 0$, we can take $\Gamma_n = 0, \ n \geq 2$, if we define f_n as

$$f_2 = -\frac{\gamma_2}{\gamma_1}, \ f_3 = -\frac{\gamma_3 + 2\gamma_2 f_2}{\gamma_1} = -\frac{\gamma_3 - 2\gamma_2^2/\gamma_1}{\gamma_1}, ... \quad (34)$$

So, we get the exact value for the anomalous dimension

$$\gamma(A) = \gamma_1 A = \gamma_1 f^{-1}(a) = \gamma_1(a + \gamma_2/\gamma_1 a^2 + \gamma_3/\gamma_1 a^3 + ... :) \quad (35)$$

While it has been well established in the perturbative regime at high energies, QCD still lacks a comprehensive solution at low and intermediate energies, even 40 years after its invention. In order to deal with the wealth of non-perturbative phenomena, various approaches are followed with limited validity and applicability. This is especially also true for lattice QCD, various functional methods, or chiral perturbation theory, to name only a few. In neither one of these approaches the full dynamical content of QCD can yet be included. Basically, the difficulties are associated with a relativistically covariant treatment of confinement and the spontaneous breaking of chiral symmetry, the latter being a well-established property of QCD at low and intermediate energies. As a result, most hadron reactions, like resonance excitations, strong and electroweak decays etc., are nowadays only amenable to models of QCD. Most famous is the constituent-quark model (CQM, 1964), which essentially relies on a limited number of effective degrees of freedom with the aim of encoding the essential features of low- and intermediate-energy QCD.

QCD, parton model, valence quarks and $\alpha_s = 2$

The CQM has a long history, and it has made important contributions to the understanding of many hadron properties, think only of the fact that the systematization of hadrons in the standard particle-data base follows the valence-quark picture. Namely the Q dependence of the nucleon form factor corresponds to three-constituent picture of the nucleon and is well described by the simple equation [Brodsky, Farrar,1973], [Matveev, Muradyan,Tavkhelidze,1973]

$$F(Q^2) \sim (Q^2)^{-2} \tag{36}$$

It was noted [Voloshin, Ter-Martyrosian, 1984] that parton densities given by the following solution

$$\begin{aligned}
M_2(Q^2) &= \frac{3}{25} + \frac{2}{3}\omega^{32/81} + \frac{16}{75}\omega^{50/81}, \\
\bar{M}_2(Q^2) &= M_2^s(Q^2) = \frac{3}{25} - \frac{1}{3}\omega^{32/81} + \frac{16}{75}\omega^{50/81}, \\
M_2^G(Q^2) &= \frac{16}{25}(1 - \omega^{50/81}), \\
\omega &= \frac{\alpha_s(Q^2)}{\alpha_s(m^2)},\ Q^2 \in (5, 20)GeV^2,\ b = 9,\ \alpha_s(Q^2) \simeq 0.2
\end{aligned} \tag{37}$$

of the Altarelli-Parisi equation

$$\dot{M} = AM,\ M^T = (M_2, \bar{M}_2, M_2^s, M_2^G),$$
$$M_2 = \int_0^1 dxx(u(x)+d(x)),\ \bar{M}_2 = \int_0^1 dxx(\bar{u}(x)+\bar{d}(x)),$$
$$M_2^s = \int_0^1 dxx(s(x)+\bar{s}(x)),\ M_2^G = \int_0^1 dxxG(x),\ \dot{M} = Q^2\frac{dM}{dQ^2}$$
$$A = -a(Q^2)\begin{pmatrix} 32/9 & 0 & 0 & -2/3 \\ 0 & 32/9 & 0 & -2/3 \\ 0 & 0 & 32/9 & -2/3 \\ -32/9 & -32/9 & -32/9 & 2 \end{pmatrix},\ a = (\frac{g}{4\pi})^2 \quad (38)$$

with the following "valence quark" initial condition at a scale m

$$M_2(m^2) = 1,\ \bar{M}_2 = M_2^s = M_2^G(m^2) = 0,\ \alpha_s(m^2) = 2 \quad (39)$$

gives the experimental values

$$M_2 = 0.44,\ \bar{M}_2 = M_2^s = 0.04,\ M_2^G = 0.48 \quad (40)$$

So, for valence quark model (VQCD), $\alpha_s(m^2) = 2$. We have seen, that for $\pi\rho N$ model $\alpha_{\pi\rho N} = 3$, and for πN model $\alpha_{\pi N} = 13$. It is nice that $\alpha_s^2 + \alpha_{\pi\rho N}^2 = \alpha_{\pi N}$. This relation can be seen, e.g., by considering pion propagator in the low energy πN model and in superposition of higher energy VQCD and $\pi\rho N$ models.
Note that to $\alpha_s = 2$ corresponds

$$g = \sqrt{4\pi\alpha_s} = 5.013 = 5+ \tag{41}$$

The AdS/CFT duality provides a gravity description in a (d + 1)-dimensional AdS space-time in terms of a flat d-dimensional conformally-invariant quantum field theory defined at the AdS asymptotic boundary
[Maldacena, 1999],[Gubser,Klebanov,Polyakov, 1998],[Witten, 1998]. Thus, in principle, one can compute physical observables in a strongly coupled gauge theory in terms of a classical gravity theory. The β-function for the nonperturbative effective coupling obtained from the LF holographic mapping in a positive dilaton modified AdS background is
[Brodsky, de Tèramond, Deur, 2010]

$$\beta(\alpha_{AdS}) = \frac{d\alpha_{AdS}}{\ln Q^2} = -\frac{Q^2}{4k^2}\alpha_{AdS}(Q^2) = \alpha_{AdS}(Q^2)\ln\frac{\alpha_{AdS}(Q^2)}{\alpha(0)} \leq 0 \quad (42)$$

where the physical QCD running coupling in its nonperturbative domain is

$$\alpha_{AdS}(Q^2) = \alpha(0)e^{-Q^2/4k^2} \quad (43)$$

So, this renormdynamics of QCD interpolates between IR fixed point $\alpha(0)$, which we take as $\alpha(0) = 2$, and UV fixed point $\alpha(\infty) = 0$.

For the QCD running coupling [Diakonov, 2003]

$$\alpha(q^2) = \frac{4\pi}{9\ln(\frac{q^2+m_g^2}{\Lambda^2})} \tag{44}$$

where $m_g = 0.88GeV,\ \Lambda = 0.28GeV$, the $\beta-$function of renormdynamics is

$$\beta(q^2) = -\frac{\alpha^2}{k}(1 - c\exp(-\frac{k}{\alpha})),$$
$$k = \frac{4\pi}{9} = 1.40,\ c = \frac{m_g^2}{\Lambda^2} = (3.143)^2 = 9.88 \tag{45}$$

for nontrivial (IR) fixed point we have

$$\alpha_{IR} = \frac{k}{\ln c} = 0.61 \tag{46}$$

For $\alpha(0) = 2$, we predict the gluon mass as

$$m_g = \Lambda e^{\frac{k}{2\alpha(0)}} = 1.42\Lambda = m_N/3,\ \Lambda = 220MeV. \tag{47}$$

The ghost-gluon interaction in Landau gauge has been determined either from DSEs [Zwanziger, 2002],[Lerche,von Smekal, 2002], or the Exact Renormalization Group Equations (ERGEs) [Pawlowski et al, 2004],[Fischer,Gies, 2004] and yield an IR fixed point

$$\alpha(0) = \frac{2\pi}{3N_c}\frac{\Gamma(3-2k)\Gamma(3+k)\Gamma(1+k)}{\Gamma(2-k)^2\Gamma(2k)} = \frac{8.9115}{N_c} = 2.970,$$
$$N_c = 3,\ k = (93-\sqrt{1201})/98 = 0.5954 \quad (48)$$

Note that, from this formula for $k = 0.6036$ we have $\alpha(0) = 3$ and for $k = 0.36$ we have $\alpha(0) = 2$.

The mane motion equation of the renormdynamics

$$\dot{a} = \beta_a \tag{49}$$

has fixed points a_c in the zeros of the $\beta_a = \beta(a_c) = 0$. At these points corresponding field theory is scale and conformal symmetric. By reparametrization $a = f(A)$, we can change the form of the motion equation and particulary we can take the minimal form of the β- functions depending only on the reparametrization invariant coefficients, e.g. for QCD in critical $d = 4$ dimensions

$$\dot{a} = \beta_2 a^2 + \beta_3 a^2, \tag{50}$$

This case, we have the trivial zero $a_c = 0$, corresponding to the scale and conformal symmetry of QCD at small scales (Higher energies). There are an opinion that at low energy we have another, the nontrivial fixed point. Personally my believe is that the fixed point is $\alpha_s(M) = 2$ at the valence quark scale $M \sim 300 MeV$. But it is obvious that the minimal form of the QCD renormdynamics (50) has not the finite nontrivial fixed point! How I can talk about the fixed point?

Thing is that, the original (complete, physical, if you like) β- function and the minimal one are connected as

$$\beta_a = f'(A)\beta_A, \tag{51}$$

so, when the minimal β- function has not the nontrivial fixed point-zero, that fixed point is given by critical point of the reparametrization function, $f(A),\ f'(A_c) = 0$. Then, when the minimal β- function has not the nontrivial zero, but we know somehow the fixed point, we can consider by corresponding reparametrization a next to the minimal forms of the β-function which will have the nontrivial fixed point.
If we do not know the value of the nontrivial fixed point, we can find its approximation value from the zeros of the reparametrization function $f(A)$, which reduce known approximation value of the β- function to the minimal one.

The renormdynamic properties of Quantum Chromodynamics were the reason of acceptance of this theory as the theory of strong interactions. The central role played by the QCD β-function, calculated at the one-['t Hooft, 1972],[Gross, Wilczek, 1973],[Politzer, 1973], two-[Caswell, 1974],[Jones,1974], [Egorian,Tarasov, 1979], three-[Tarasov,Vladimirov,Zharkov,1980],[Larin,Vermaseren,1993] and finally at the four-loop [van Ritbergen,Vermaseren,Larin,1997] level, cannot be overestimated in this respect.
The minimal form of the QCD renormdynamics (RD) is

$$\dot{x} = -b_2 x^2 - b_3 x^3,$$
$$b_2 = 11 - \frac{2}{3}n,\ b_3 = 2(51 - \frac{19}{3}n),\ x = \frac{\alpha_s}{4\pi} = (\frac{g}{4\pi})^2, \qquad (52)$$

where n is the number of the light quarks,e.g. $n = 3$ for energy scales less then the mass of the $c-$quark, $m_c \simeq 1GeV$ but higher than the mass of $s-$ quark, $m_s \simeq 100MeV$.

The Bjorken sum rule [Bjorken,1966] has been of central importance for studying the spin structure of the nucleon. Accounting for finite Q^2 corrections to the sum rule, it reads:

$$\int_0^1 (g_{p1} - g_{n1})dx = \frac{g_A}{6}(1 - \frac{\alpha_s}{\pi} - 3.58(\frac{\alpha_s}{\pi})^2 - 20.21(\frac{\alpha_s}{\pi})^3 + ...) + \sum_{k\geq 1} \frac{\mu_k}{Q^{2k}} \quad (53)$$

where the μ_k are higher twist terms. We take the following valence quark parametrization of the α_s

$$\int_0^1 (g_{p1} - g_{n1})dx = \frac{g_A}{6}(1 - \frac{\alpha_V}{2}),\ \alpha_V = \frac{2\alpha_s}{\pi} + ... \quad (54)$$

The Bjorken sum rule is related to a more general sum rule, the generalized Gerasimov-Drell-Hearn (GDH) sum rule
[Gerasimov, 1965],[Drell, Hearn, 1966],
[Anselmino,Ioffe,Leader,1989],[Ji,Osborne,2001]:

$$\int_0^1 (g_{p1} - g_{n1})dx = \frac{Q^2}{(4\pi)^2\alpha}(GDH_p(Q^2) - GDH_n(Q^2)) \quad (55)$$

Since the generalized GDH sum is, in principle, calculable at any Q^2, it can be used to study the transition from the partonic to hadronic degrees of freedom of the strong force. The Bjorken sum is the flavor non-singlet part of the GDH sum. This leads to simplifications that may help in linking the (χPT) validity domain to the pQCD validity domain [Burkert,2001]. Hence the Bjorken sum appears as a key quantity to study the hadron-parton transition.

It is sixty years since Yang and Mills (1954) performed their pioneering work on gauge theories, and we have in our hands a good candidate for a theory of the strong interactions based, precisely, on a non-Abelian gauge theory, QCD.
We considered the main properties of the renormdynamics, corresponding motion equations and their solutions on the examples of QCD and other field theory models.

With the advent of any new hadron accelerator the quantities first studied are charged particle multiplicities. The multiparticle production can be described by the probability distribution P_n which is a superposition of some unknown distribution of sources F, and the Poisson distribution describing particle emission from one source. This is a typical situation in many microscopic models of multiparticle production.
Independently radiating valence quarks and corresponding negative binomial distribution presents phenomenologically preferable mechanism of hadronization in multiparticle production processes.

Let us consider $l-$particle semi-inclusive distribution

$$F_l(n,q)=\frac{d^l\sigma_n}{\bar{d}q_1...\bar{d}q_l}=\frac{1}{n!}\int\prod_{i=1}^{n}\bar{d}q_i'\delta(p_1+p_2-\Sigma_{i=1}^{l}q_i-\Sigma_{i=1}^{n}q_i')$$
$$\cdot|M_{n+l+2}(p_1,p_2,q_1,...,q_l,q_1',...,q_n';g(\mu),m(\mu)),\mu)|^2,$$
$$\bar{d}p\equiv\frac{d^3p}{E(p)},\ E(p)=\sqrt{p^2+m^2}. \quad (56)$$

From the renormdynamic equation

$$DM_{n+l+2}=\frac{\gamma}{2}(n+l+2)M_{n+l+2}, \quad (57)$$

we obtain

$$
\begin{aligned}
&DF_l(n,q) = \gamma(n+l+2)F_l(n,q),\\
&DF_l(q) = \gamma(<n> + l + 2)F_l(q),\\
&D <n^k(q)> = \gamma(<n^{k+1}(q)> - <n^k(q)><n(q)>),\\
&DC_k = \gamma <n(q)> (C_{k+1} - C_k(1 + k(C_2 - 1)))\\
&F_l(q) \equiv \frac{d^l\sigma}{\bar{d}q_1 \ldots \bar{d}q_l} = \sum_n \frac{d^l\sigma_n}{\bar{d}q_1 \ldots \bar{d}q_l},\ <n^k(q)> = \frac{\sum_n n^k d^l\sigma_n/\bar{d}q^l}{\sum_n d^l\sigma_n/\bar{d}q^l}\\
&C_k = \frac{<n^k(q)>}{<n(q)>^k}
\end{aligned} \tag{58}
$$

From dimensional considerations, the following combination of cross sections [Koba et al, 1972] must be universal function

$$< n > \frac{\sigma_n}{\sigma} = \Psi(\frac{n}{< n >}). \quad (59)$$

Corresponding relation for the inclusive cross sections is [Matveev et al, 1976].

$$< n(p) > \frac{d\sigma_n}{d\bar{p}} / \frac{d\sigma}{d\bar{p}} = \Psi(\frac{n}{< n(p) >}). \quad (60)$$

Indeed, let us define $n-$dimension of observables [Makhaldiani, 1980]

$$[n] = 1,\ [\sigma_n] = -1, \sigma = \Sigma_n \sigma_n,\ [\sigma] = 0,\ [< n >] = 1. \quad (61)$$

The following expression does not depend on any dimensional quantities and must have a corresponding universal form

$$P_n =< n > \frac{\sigma_n}{\sigma} = \Psi(\frac{n}{< n >}). \quad (62)$$

For any discrete variable n, if the change of summation on the integration is good approximation, we can invent corresponding dimension and use dimensional counting.

Let us find an explicit form of the universal functions using renormdynamic equations. From the definition of the moments we have

$$C_k = \int_0^\infty dx x^k \Psi(x), \tag{63}$$

so they are universal parameters,

$$DC_k = 0 \Rightarrow C_{k+1} = (1 + k(C_2 - 1))C_k \Rightarrow$$
$$C_k = (1 + (k-1)(C_2 - 1))...(1 + 2(C_2 - 1))C_2. \tag{64}$$

Now we can invert momentum transform and find (see [Makhaldiani, 1980]) universal functions [Ernst, Schmit, 1976], [Darbaidze et al, 1978].

$$\Psi(z) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dn z^{-n-1} C_n = \frac{c^c}{\Gamma(c)} z^{c-1} e^{-cz},$$
$$C_2 = 1 + \frac{1}{c} \tag{65}$$

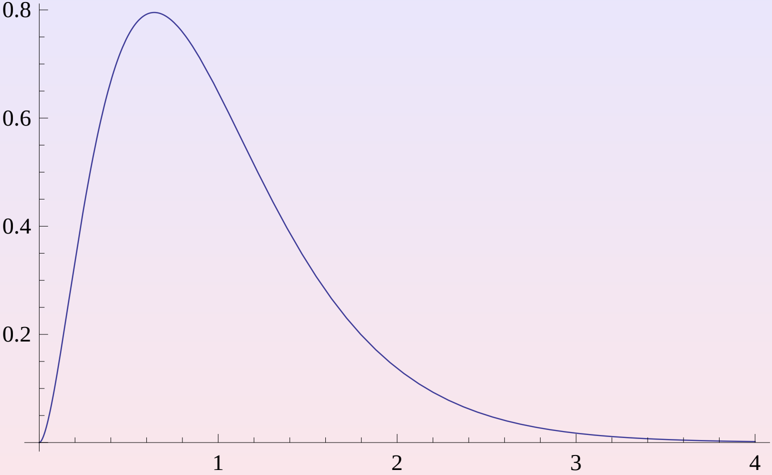

Figure: KNO distribution, $\Psi(z)$, with $c = 2.8$

The value of the parameter c can be measured from the dispersion low,

$$D = \sqrt{< n^2 > - < n >^2} = \sqrt{C_2 - 1} < n >= A < n >,$$
$$A = \frac{1}{\sqrt{c}} \simeq 0.6, \ \ c = 2.8;$$
$$(c = 3, \ \ A = 0.58) \quad (66)$$

which is in accordance with $n-$dimension counting.

We can calculate also $1/<n>$ correction to the scaling function

$$<n>\frac{\sigma_n}{\sigma} = \Psi = \Psi_0(\frac{n}{<n>}) + \frac{1}{<n>}\Psi_1(\frac{n}{<n>}),$$
$$C_k = C_k^0 + \frac{1}{<n>}C_k^1,$$
$$C_k^0 = \int_0^\infty dx x^k \Psi_0(x),\ C_k^1 = \int_0^\infty dx x^k \Psi_1(x),$$
$$\Psi_1(z) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dn z^{-n-1} C_n^1 = \frac{C_2^1 c^2}{2}(z - 2 + \frac{c-1}{cz})\Psi_0 \quad (67)$$

The characteristic function we define as

$$\Phi(t) = \int_0^\infty dx e^{tx} \Psi(x) = (1 - t/c)^{-c}, \ Re(t) < c \tag{68}$$

For the moments of the distribution, we have

$$\Phi^{(k)}(0) = C_k = (-c)(-c-1)...(-c-k+1)(-1/c)^k = \frac{\Gamma(c+k)}{\Gamma(c)c^k} \tag{69}$$

Note that it is an infinitely divisible characteristic function, i.e.

$$\Phi(t) = (\Phi_n(t))^n, \ \Phi_n(t) = (1 - t/c)^{-c/n} \tag{70}$$

If we calculate observable(mean) value of x, we find

$$\begin{aligned} &< x >= \Phi'(0) = n\Phi(0)_n{}' = n < x >_n, \\ &< x >_n = \frac{< x >}{n} \end{aligned} \tag{71}$$

For the second moment and dispersion, we have

$$\begin{aligned}
&< x^2 >= \Phi^{(2)}(0) = n < x^2 >_n + n(n-1) < x >_n^2, \\
&\mathrm{D}^2 = < x^2 > - < x >^2 = n(< x^2 >_n - < x >_n^2) = n\mathrm{D}_n^2 \\
&\mathrm{D}_n^2 = \frac{\mathrm{D}^2}{n} = \frac{\mathrm{D}^2}{< x >} < x >_n
\end{aligned} \tag{72}$$

In a sense, any Hamiltonian quantum (and classical) system can be described by infinitely divisible distributions, because in the functional integral formulation, we use the following step

$$U(t) = e^{-itH} = (e^{-i\frac{t}{N}H})^N \tag{73}$$

In the case of scalar field theory

$$L(\varphi) = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{g}{n}\varphi^n$$
$$= g^{\frac{2}{2-n}}(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{1}{n}\phi^n) \quad (74)$$

so, to the constituent field ϕ_N corresponds higher value of the coupling constant,

$$g_N = gN^{\frac{n-2}{2}} \quad (75)$$

For weak nonlinearity, $n = 2 + 2\varepsilon, d = 2/\varepsilon + 2, g_N = g(1 + \varepsilon \ln N + O(\varepsilon^2))$

Negative Binomial Distribution

Negative binomial distribution (NBD) is defined as

$$P(n) = \frac{\Gamma(n+r)}{n!\Gamma(r)} p^n (1-p)^r, \ \sum_{n\geq 0} P(n) = 1, \tag{76}$$

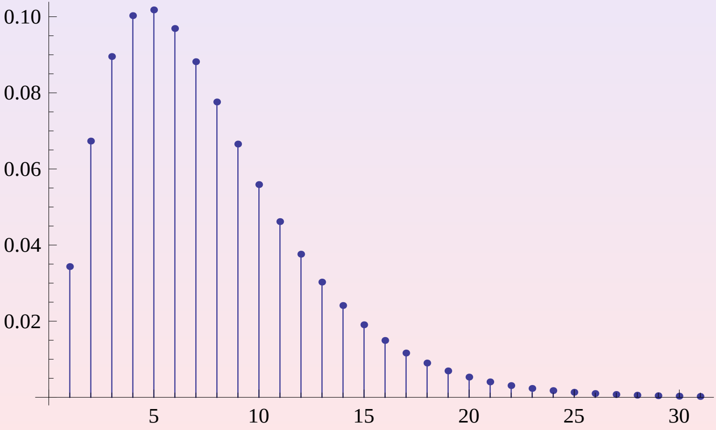

Figure: $P(n), r = 2.8, p = 0.3, < n >= 6$

NBD provides a very good parametrization for multiplicity distributions in e^+e^- annihilation; in deep inelastic lepton scattering; in proton-proton collisions; in proton-nucleus scattering.

Hadronic collisions at high energies (LHC) lead to charged multiplicity distributions whose shapes are well fitted by a single NBD in fixed intervals of central (pseudo)rapidity η [ALICE, 2010].

It is interesting to understand how NBD fits such a different reactions?

Let us consider NBD for normed topological cross sections

$$\frac{\sigma_n}{\sigma} = P(n) = \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(\frac{k}{<n>})^k(1+\frac{k}{<n>})^{-(n+k)}$$
$$= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(1+\frac{k}{<n>})^{-n}(1+\frac{<n>}{k})^{-k}$$
$$= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(\frac{<n>}{<n>+k})^n(\frac{k}{k+<n>})^k,$$
$$= \frac{\Gamma(k+n)}{\Gamma(k)n!}\frac{(\frac{k}{<n>})^k}{(1+\frac{k}{<n>})^{k+n}},$$
$$r = k > 0,\ p = \frac{<n>}{<n>+k}. \tag{77}$$

The generating function for NBD is

$$F(h) = (1+\frac{<n>}{k}(1-h))^{-k} = (1+\frac{<n>}{k})^{-k}(1-ah))^{-k},$$
$$a = p = \frac{<n>}{<n>+k}. \tag{78}$$

Indeed,

$$
\begin{aligned}
(1-ah))^{-k} &= \frac{1}{\Gamma(k)}\int_0^\infty dt t^{k-1} e^{-t(1-ah)} \\
&= \frac{1}{\Gamma(k)}\int_0^\infty dt t^{k-1} e^{-t} \sum_0^\infty \frac{(tah)^n}{n!} \\
&= \sum_0^\infty \frac{\Gamma(n+k)a^n}{\Gamma(k)n!} h^n, \\
P(n) &= (1+\frac{<n>}{k})^{-k}\frac{\Gamma(n+k)}{\Gamma(k)n!}(\frac{<n>}{<n>+k})^n \\
&= \frac{k^k\Gamma(n+k)}{\Gamma(k)\Gamma(n+1)}(<n>+k)^{-(n+k)} <n>^n \\
&= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(\frac{k}{<n>})^k(1+\frac{k}{<n>})^{-(n+k)} \qquad (79)
\end{aligned}
$$

The Bose-Einstein distribution is a special case of NBD with $k = 1$.

If k is negative, the NBD becomes a positive binomial distribution, narrower than Poisson (corresponding to negative correlations).
For negative (integer) values of $k = -N$, we have Binomial GF

$$F_{bd} = (1 + \frac{<n>}{N}(h-1))^N = (a+bh)^N,\ a = 1 - \frac{<n>}{N}, b = \frac{<n>}{N},$$
$$P_{bd}(n) = C_N^n(\frac{<n>}{N})^n(1 - \frac{<n>}{N})^{N-n} \tag{80}$$

(In a sense) we have a (quantum) spectrum for the parameter k, which contains any (positive) real values and (with finite number of states) the negative integer values, $(0 \leq n \leq N)$
From the generating function we have

$$<n^2>= (\frac{hd}{dh})^2 F(h)|_{h=1} = \frac{k+1}{k} <n>^2 + <n>, \tag{81}$$

for dispersion we obtain

$$D = \sqrt{<n^2> - <n>^2} = \frac{1}{\sqrt{k}} <n> (1 + \frac{k}{<n>})^{1/2}$$
$$= \frac{1}{\sqrt{k}} <n> + \frac{\sqrt{k}}{2} + O(1/<n>), \tag{82}$$

So, the dispersion low for KNO and NBD distributions are the same, with $c = k$, for high values of the mean multiplicity.
The factorial moments of NBD,

$$F_m = (\frac{d}{dh})^m F(h)|_{h=1} = \frac{< n(n-1)...(n-m+1) >}{< n >^m} = \frac{\Gamma(m+k)}{\Gamma(m)k^m}, \quad (83)$$

and usual normalized moments of KNO (69) coincides.

Using fractal calculus (see e.g. [Makhaldiani, 2003]),

$$
\begin{aligned}
D_{0,x}^{-\alpha} f &= \frac{|x|^\alpha}{\Gamma(\alpha)} \int_0^1 |1-t|^{\alpha-1} f(xt)dt, = \frac{|x|^\alpha}{\Gamma(\alpha)} B(\alpha,\ \partial x) f(x) \\
&= |x|^\alpha \frac{\Gamma(\partial x)}{\Gamma(\alpha+\partial x)} f(x),\ f(xt) = t^{x\frac{d}{dx}} f(x). \qquad (84)
\end{aligned}
$$

we can define factorial and cumulant moments for any complex indexes,

$$
\begin{aligned}
F_{-q} &=< n >^q D_{0,x}^{-q} G_{NBD}(x)|_{x=0} = \frac{k^q \Gamma(k-q)}{\Gamma(k)}, \\
K_{-q} &=< n >^q D_{0,x}^{-q} \ln G_{NBD}(x)|_{x=0} = k^{q+1}\Gamma(-q), \\
H_{-q} &= \frac{\Gamma(k+1)\Gamma(-q)}{\Gamma(k-q)} \qquad (85)
\end{aligned}
$$

Let us show that NBD is a discrete distribution corresponding to the KNO scaling,

$$\lim_{<n>\to\infty} <n> P_n|_{\frac{n}{<n>}=z} = \Psi(z) \quad (86)$$

Indeed, using the following asymptotic formula

$$\Gamma(x+1) = x^x e^{-x}\sqrt{2\pi x}(1+\frac{1}{12x}+O(x^{-2})), \quad (87)$$

we find

$$<n> P_n =<n> \frac{(n+k-1)^{n+k-1}e^{-(n+k-1)}}{\Gamma(k)n^n e^{-n}}\frac{k^k}{n^k} <n> z^k e^{-k\frac{n+k}{<n>}}$$
$$= \frac{k^k}{\Gamma(k)} z^{k-1}e^{-kz} + O(1/<n>) \quad (88)$$

We can calculate also $1/<n>$ correction term to the KNO from the NBD. The answer is

$$\Psi = \frac{k^k}{\Gamma(k)} z^{k-1} e^{-kz}(1+\frac{k^2}{2}(z-2+\frac{k-1}{kz})\frac{1}{<n>}) \quad (89)$$

This form coincides with the corrected KNO (67) for $c = k$ and $C_2^1 = 1$. We have seen that KNO characteristic function (68) and NBD GF (78) have almost same form. This relation become in coincidence if

$$c = k,\ t = (h-1)\frac{<n>}{k} \tag{90}$$

Now the definition of the characteristic function (68) can be read as

$$\int_0^\infty e^{-<n>z(1-h)}\Psi(z)dz = (1 + \frac{<n>}{k}(1-h))^{-k} \tag{91}$$

which means that Poisson GF weighted by KNO distribution gives NBD GF. Because of this, the NBD is the gamma-Poisson (mixture) distribution. This is the exact and universal picture of hadronization in multiparticle production processes.

Fore high values of $x_2 = k$ the NBD distribution reduces to the Poisson distribution

$$F(x_1, x_2, h) = (1 + \frac{x_1}{x_2}(1-h))^{-x_2} \Rightarrow e^{-x_1(1-h)} = e^{-<n>} e^{h<n>}$$
$$= \sum P(n) h^n,$$
$$P(n) = e^{-<n>} \frac{<n>^n}{n!} \tag{92}$$

For the Poisson distribution

$$\frac{d^2F(h)}{dh^2}|_{h=1} =< n(n-1) >=< n >^2,$$
$$D^2 =< n^2 > - < n >^2 =< n > . \tag{93}$$

In the case of NBD, we had the following dispersion low

$$D^2 = \frac{1}{k} < n >^2 + < n >, \tag{94}$$

which coincides with the previous expression for high values of k.

Poisson GF belongs to the class of the infinitely divisible distributions,

$$F(h, < n >) = (F(h, < n > /k))^k \tag{95}$$

For high values of $< n >$, the Poisson distribution reduces to the Gauss distribution

$$P(n) = e^{-<n>} \frac{< n >^n}{n!} \Rightarrow \frac{1}{\sqrt{2\pi < n >}} \exp(-\frac{(n- < n >)^2}{2 < n >}) \tag{96}$$

For high values of k in the integral relation (91), in the KNO function dominates the value $z_c = 1$ and both sides of the relation reduce to the Poisson GF.

A Bose-Einstein, or geometrical, distribution is a thermal distribution for single state systems. An useful property of the negative binomial distribution with parameters

$$< n >,\ k$$

is that it is (also) the distribution of a sum of k independent random variables drawn from a Bose-Einstein distribution with mean $< n > /k$,

$$\begin{aligned} P_n &= \frac{1}{< n > +1}\left(\frac{< n >}{< n > +1}\right)^n \\ &= (e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2})e^{-\beta\hbar\omega(n+1/2)},\ T = \frac{\hbar\omega}{\ln\frac{<n>+1}{<n>}} \\ &\sum_{n\geq 0} P_n = 1,\ \sum nP_n =< n >= \frac{1}{e^{\beta\hbar\omega - 1}},\ T \simeq \hbar\omega < n >,\ < n > \gg 1, \\ P(x) &= \sum_n x^n P_n = (1+ < n > (1-x))^{-1}. \end{aligned} \tag{97}$$

This is easily seen from the generating function in (78), remembering that the generating function of a sum of independent random variables is the product of their generating functions.

Indeed, for

$$n = n_1 + n_2 + ... + n_k, \quad (98)$$

with n_i independent of each other, the probability distribution of n is

$$P_n = \sum_{n_1,...,n_k} \delta(n - \sum n_i) p_{n_1} ... p_{n_k},$$
$$P(x) = \sum_n x^n P_n = p(x)^k \quad (99)$$

This has a consequence that an incoherent superposition of N emitters that have a negative binomial distribution with parameters $k, < n >$ produces a negative binomial distribution with parameters $Nk, N < n >$.
So, for the GF of NBD we have (N=2)

$$F(k, < n >) F(k, < n >) = F(2k, 2 < n >) \quad (100)$$

And more general formula (N=m) is

$$F(k, < n >)^m = F(mk, m < n >) \quad (101)$$

We can put this equation in the closed nonlocal form

$$Q_q F = F^q, \tag{102}$$

where

$$Q_q = q^D, \quad D = \frac{kd}{dk} + \frac{< n > d}{d < n >} = \frac{x_1 d}{dx_1} + \frac{x_2 d}{dx_2} \tag{103}$$

Note that temperature defined in (97) gives an estimation of the Glukvar temperature when it radiates hadrons. If we take $\hbar\omega = 100MeV$, to $T \simeq T_c \simeq 200MeV$ corresponds $< n > \simeq 1.5$ If we take $\hbar\omega = 10MeV$, to $T \simeq T_c \simeq 200MeV$ corresponds $< n > \simeq 20$. A singular behavior of $< n >$ may indicate corresponding phase transition and temperature. At that point we estimate characteristic quantum $\hbar\omega$.
We see that universality of NBD in hadron-production is similar to the universality of black body radiation.

Let us imagine space-time development of the the multiparticle process and try to describe it by some (phenomenological) dynamical equation. We start to find the equation for the Poisson distribution and than naturally extend them for the NBD case.
Let us define an integer valued variable $n(t)$ as a number of events (produced particles) at the time $t,\ n(0) = 0$. The probability of event $n(t),\ P(t, n)$, is defined from the following motion equation

$$
\begin{aligned}
&P_t \equiv \frac{\partial P(t,n)}{\partial t} = r(P(t,n-1) - P(t,n)),\ n \geq 1 \\
&P_t(t,0)) = -rP(t,0), \\
&P(t,n) = 0,\ n < 0,
\end{aligned}
\tag{104}
$$

so

$$
\begin{aligned}
&P(t,0) \equiv P_0(t) = e^{-rt}, \\
&P(t,n) = Q(t,n)P_0(t), \\
&Q_t(t,n) = rQ(t,n-1),\ Q(t,0) = 1.
\end{aligned}
\tag{105}
$$

To solve the equation for Q, we invent its generating function

$$
F(t,h) = \sum_{n>0} h^n Q(t,n),
\tag{106}
$$

and solve corresponding equation

$$F_t = rhF,\ F(t,h) = e^{rth} = \sum h^n \frac{(rt)^n}{n!},\ Q(t,n) = \frac{(rt)^n}{n!}, \tag{107}$$

so

$$P(t,n) = e^{-rt} \frac{(rt)^n}{n!} \tag{108}$$

is the Poisson distribution.
If we compare this distribution with (96), we identify $< n >= rt$, as if we have a free particle motion with velocity r and the distance is the mean multiplicity. This way we have a connection between n-dimension of the multiplicity and the usual dimension of trajectory.
As the equation gives right solution, its generalization may give more general distribution, so we will generalize the equation (104). For this, we put the equation in the closed form

$$\begin{aligned} P_t(t,n) &= r(e^{-\partial_n} - 1)P(t,n) \\ &= \sum_{k\geq 1} D_k \partial^k P(t,n),\ D_k = (-1)^k \frac{r}{k!}, \end{aligned} \tag{109}$$

where the $D_k,\ k \geq 1$, are generalized diffusion coefficients.

For other values of the coefficients, we will have other distributions. For mean square deviation of the trajectory we have

$$< (x - \bar{x})^2 >=< x^2 > - < x >^2 \equiv D(x)^2 \sim t^{2/d_f}, \quad (110)$$

where d_f is fractal dimension. For smooth classical trajectory of particles we have $d_f = 1$; for free stochastic, Brownian, trajectory, all diffusion coefficients are zero but D_2, we have $d_f = 2$. In the case of Poisson process we have,

$$D(n)^2 =< n^2 > - < n >^2 \sim t, \; d_f = 2. \quad (111)$$

In the case of the NBD and KNO distributions

$$D(n)^2 \sim t^2, \; d_f = 1. \quad (112)$$

As we have seen, rasing k, KNO reduce to the Poisson, so we have a dimensional (phase) transition from the phase with dimension 1 to the phase with dimension 2. It is interesting, if somehow this phase transition is connected to the other phase transitions in strong interaction processes.

For the Poisson distribution GF is solution of the following equation,

$$\dot{F} = -r(1-h)F, \tag{113}$$

For the NBD corresponding equation is

$$\dot{F} = \frac{-r(1-h)}{1+\frac{rt}{k}(1-h)}F = -R(t)F,\ R(t) = \frac{r(1-h)}{1+\frac{rt}{k}(1-h)}. \tag{114}$$

If we change the time variable as $t = T^{d_f}$, we reduce the dispersion low from general fractal to the NBD like case. Corresponding transformation for the evolution equation is

$$F_T = -d_f T^{d_f-1} R(T^{d_F})F, \tag{115}$$

we ask that this equation coincides with NBD motion equation, and define rate function $R(T)$

$$d_f T^{d_f-1} R(T^{d_F}) = \frac{r(1-h)}{1+\frac{rT}{k}(1-h)} \tag{116}$$

The following equation defines a production processes with fractal dimension d_F

$$F_t = -R(t)F,\ R(t) = \frac{r(1-h)}{d_F t^{\frac{d_F-1}{d_F}}(1+\frac{rt^{1/d_F}}{k}(1-h))} \tag{117}$$

Motion equations of physics (applied mathematics in general) connect different observable quantities and reduce the number of independently measurable quantities. More fundamental equation contains less number of independent quantities. When (before) we solve the equations, we invent dimensionless invariant variables, than one solution can describe all of the class of phenomena.
In the z - Scaling (zS) approach to the inclusive multiparticle distributions (MPD) (see, e.g. [Tokarev, Zborovsky, 2007]), different inclusive distributions depending on the variables $x_1, ...x_n$, are described by universal function $\Psi(z)$ of fractal variable z,

$$z = x_1^{-\alpha_1} ... x_n^{-\alpha_n}. \tag{118}$$

It is interesting to find a dynamical system which generates this distributions and describes corresponding MPD.

We can find a good function if we know its derivative. Let us consider the following RD like equation

$$z\frac{d}{dz}\Psi = V(\Psi),$$
$$\int_{\Psi(z_0)}^{\Psi(z)} \frac{dx}{V(x)} = \ln\frac{z}{z_0} \tag{119}$$

As a dimensionless physical quantity $\Psi(z)$ may depend only on the running coupling constant $g(\tau),\ \tau = \ln z/z_0$

$$z\frac{d}{dz}\Psi = \dot{\Psi} = \frac{d\Psi}{dg}\beta(g) = U(g) = U(f^{-1}(\Psi)) = V(\Psi),$$
$$\Psi(\tau) = f(g(\tau)),\ g = f^{-1}(\Psi(\tau)) \tag{120}$$

Realistic solution for Ψ

According to the paper [Tokarev, Zborovsky, 2007], for high values of z, $\Psi(z) \sim z^{-\beta}$; for small z, $\Psi(z) \sim$ const.
So, for high z,

$$z\frac{d}{dz}\Psi = V(\Psi(z)) = -\beta\Psi(z); \quad (121)$$

for smaller values of z, $\Psi(z)$ rise and we expect nonlinear terms in $V(\Psi)$,

$$V(\Psi) = -\beta\Psi + \gamma\Psi^2. \quad (122)$$

With this function, we can solve the equation for Ψ and find

$$\Psi(z) = \frac{1}{\frac{\gamma}{\beta} + cz^{\beta}}. \quad (123)$$

Let us consider more general potential V

$$z\frac{d}{dz}\Psi = V(\Psi) = -\beta\Psi(z) + \gamma\Psi(z)^{1+n} \tag{124}$$

Corresponding solution for Ψ is

$$\Psi(z) = \frac{1}{(\frac{\gamma}{\beta} + cz^{n\beta})^{\frac{1}{n}}} \tag{125}$$

More general solution contains three parameters and may better describe the data of inclusive distributions.

More general solution for Ψ

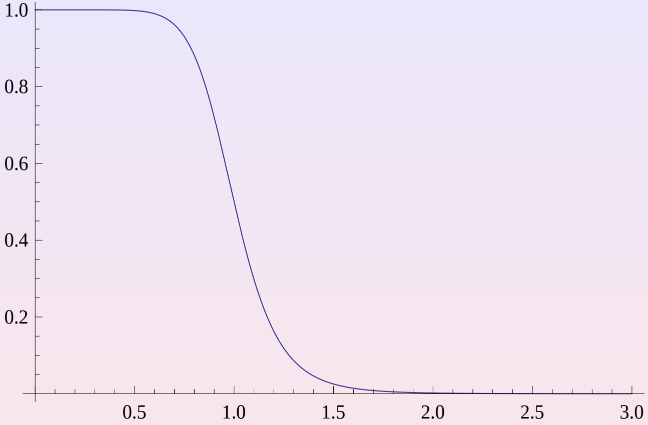

Figure: z-scaling distribution $\Psi(z, 9, 9, 1, 1)$

In the case of $n = 1$ we reproduce the previous solution.
Another "natural" case is $n = 1/\beta$,

$$\Psi(z) = \frac{1}{(\frac{\gamma}{\beta} + cz)^{\beta}} \tag{126}$$

In this case, we can absorb (interpret) the combined parameter by shift and scaling

$$z \to \frac{1}{c}(z - \frac{\gamma}{\beta}) \tag{127}$$

Another interesting point of view is to predict the value of β

$$\beta = \frac{1}{n} = 0.5; 0.33; 0.25; 0.2; ...,\ \ n = 2, 3, 4, 5, ... \tag{128}$$

For experimentally suggested value $\beta \simeq 9, n = 0.11$
The three parameter function is restricted by the normalization condition

$$\int_0^{\infty} \Psi(z) dz = 1,$$

$$B(\frac{\beta - 1}{\beta n}, \frac{1}{\beta n}) = (\frac{\beta}{\gamma})^{\frac{\beta-1}{\beta n}} \frac{\beta n}{c^{\beta n}}, \tag{129}$$

When $\beta n = 1$, we have

$$c = (\beta - 1)(\frac{\beta}{\gamma})^{\beta-1} \tag{130}$$

If $\beta n = 1$ and $\beta = \gamma$, than $c = \beta - 1$.
In general

$$c^{\beta n} = (\frac{\beta}{\gamma})^{\frac{\beta-1}{\beta n}} \frac{\beta n}{B(\frac{\beta-1}{\beta n}, \frac{1}{\beta n})} \tag{131}$$

The dimension of the space(-time) is the model dependent concept. E.g. for the fundamental bosonic string model (in flat space-time) the dimension is 26; for superstring model the dimension is 10 [Kaku, 2000].
Let us imagine that we have some action-functional formulation with the fundamental motion equation

$$z\frac{d}{dz}\Psi = V(\Psi(z)) = V(\Psi) = -\beta\Psi + \gamma\Psi^{1+n}. \tag{132}$$

Than, the corresponding Lagrangian contains the following mass and interaction parts

$$-\frac{\beta}{2}\Psi^2 + \frac{\gamma}{2+n}\Psi^{2+n} \tag{133}$$

The action gives renormalizable (effective quantum field theory) model when

$$d+2 = \frac{2N}{N-2} = \frac{2(2+n)}{n} = 2 + \frac{4}{n} = 2 + 4\beta, \tag{134}$$

so, measuring the parameter β inside hadronic and nuclear matters, we find corresponding (fractal) dimension.

From fundamental equation we obtain

$$(z\frac{d}{dz})^2\Psi \equiv \ddot{\Psi} = V'(\Psi)V(\Psi) = \frac{1}{2}(V^2)'$$
$$= \beta^2\Psi - \beta\gamma(n+2)\Psi^{n+1} + \gamma^2(n+1)\Psi^{2n+1} \tag{135}$$

Corresponding action Lagrangian is

$$L = \frac{1}{2}\dot{\Psi}^2 + U(\Psi),\ U = \frac{1}{2}V^2 = \frac{1}{2}\Psi^2(\beta - \gamma\Psi^n)^2$$
$$= \frac{\beta^2}{2}\Psi^2 - \beta\gamma\Psi^{2+n} + \frac{\gamma^2}{2}\Psi^{2+2n} \tag{136}$$

This potential, $-U$, has two maximums, when $V = 0$, and minimum, when $V' = 0$, at $\Psi = 0$ and $\Psi = (\beta/\gamma)^{1/n}$, and $\Psi = (\beta/(n+1)\gamma)^{1/n}$, correspondingly.

We define time-space-scale field $\Psi(t, x, \eta)$, where $\eta = \ln z-$ is scale coordinate variable, with corresponding action functional

$$A = \int dt d^d x d\eta (\frac{1}{2}g^{ab}\partial_a\Psi\partial_b\Psi + U(\Psi)) \tag{137}$$

The renormalization constraint for this action is

$$N = 2 + 2n = \frac{2(2+d)}{2+d-2} = 2 + \frac{4}{d},\ dn = 2,\ d = 2/n = 2\beta. \tag{138}$$

So we have two models for spase-time dimension, (134) and (138),

$$d_1 = 4\beta;\ d_2 = 2\beta \tag{139}$$

The coordinate η characterise (multiparticle production) physical process at the (external) space-time point (x,t). The dimension of the space-time inside hadrons and nuclei, where multiparticle production takes place is

$$d + 1 = 1 + 2\beta \tag{140}$$

Note that this formula reminds the dimension of the spin s state, $d_s = 2s + 1$. If we take $\beta (= s) = 0; 1/2; 1; 3/2; 2; ...$ We will have $d + 1 = 1; 2; 3; 4; 5; ...$

Let us consider a general dynamical system described by the following system of the ordinary differential equations [Arnold, 1978]

$$\dot{x}_n = v_n(x),\ 1 \leq n \leq N, \tag{141}$$

$\dot{x}_n$ stands for the total derivative with respect to the parameter t. When the number of the degrees of freedom is even, and

$$v_n(x) = \varepsilon_{nm} \frac{\partial H_0}{\partial x_m},\ 1 \leq n, m \leq 2M, \tag{142}$$

the system (141) is Hamiltonian one and can be put in the form

$$\dot{x}_n = \{x_n, H_0\}_0, \tag{143}$$

where the Poisson bracket is defined as

$$\{A, B\}_0 = \varepsilon_{nm} \frac{\partial A}{\partial x_n} \frac{\partial B}{\partial x_m} = A \frac{\overleftarrow{\partial}}{\partial x_n} \varepsilon_{nm} \frac{\overrightarrow{\partial}}{\partial x_m} B, \tag{144}$$

and summation rule under repeated indices has been used.

Let us consider the following Lagrangian

$$L = (\dot{x}_n - v_n(x))\psi_n \tag{145}$$

and the corresponding equations of motion

$$\dot{x}_n = v_n(x), \dot{\psi}_n = -\frac{\partial v_m}{\partial x_n}\psi_m. \tag{146}$$

The system (146) extends the general system (141) by linear equation for the variables ψ. The extended system can be put in the Hamiltonian form [Makhaldiani, Voskresenskaya, 1997]

$$\dot{x}_n = \{x_n, H_1\}_1, \dot{\psi}_n = \{\psi_n, H_1\}_1, \tag{147}$$

where first level (order) Hamiltonian is

$$H_1 = v_n(x)\psi_n \tag{148}$$

and (first level) bracket is defined as

$$\{A, B\}_1 = A(\frac{\overleftarrow{\partial}}{\partial x_n}\frac{\overrightarrow{\partial}}{\partial \psi_n} - \frac{\overleftarrow{\partial}}{\partial \psi_n}\frac{\overrightarrow{\partial}}{\partial x_n})B. \tag{149}$$

Note that when the Grassmann grading [Berezin, 1987] of the conjugated variables x_n and ψ_n are different, the bracket (149) is known as Buttin bracket[Buttin, 1996].
In the Faddeev-Jackiw formalism [Faddeev, Jackiw, 1988] for the Hamiltonian treatment of systems defined by first-order Lagrangians, i.e. by a Lagrangian of the form

$$L = f_n(x)\dot{x}_n - H(x), \tag{150}$$

motion equations

$$f_{mn}\dot{x}_n = \frac{\partial H}{\partial x_m}, \tag{151}$$

for the regular structure function f_{mn}, can be put in the explicit hamiltonian (Poisson; Dirac) form

$$\dot{x}_n = f_{nm}^{-1}\frac{\partial H}{\partial x_m} = \{x_n, x_m\}\frac{\partial H}{\partial x_m} = \{x_n, H\}, \tag{152}$$

where the fundamental Poisson (Dirac) bracket is

$$\{x_n, x_m\} = f_{nm}^{-1},\ f_{mn} = \partial_m f_n - \partial_n f_m. \tag{153}$$

The system (146) is an important example of the first order regular hamiltonian systems. Indeed, in the new variables,

$$y_n^1 = x_n, y_n^2 = \psi_n, \tag{154}$$

lagrangian (145) takes the following first order form

$$\begin{aligned} & L = (\dot{x}_n - v_n(x))\psi_n \Rightarrow \frac{1}{2}(\dot{x}_n\psi_n - \dot{\psi}_n x_n) - v_n(x)\psi_n \\ & = \frac{1}{2}y_n^a\varepsilon^{ab}\dot{y}_n^b - H(y) \\ & = f_n^a(y)\dot{y}_n^a - H(y), f_n^a = \frac{1}{2}y_n^b\varepsilon^{ba}, H = v_n(y^1)y_n^2, \\ & f_{nm}^{ab} = \frac{\partial f_m^b}{\partial y_n^a} - \frac{\partial f_n^a}{\partial y_m^b} = \varepsilon^{ab}\delta_{nm}; \end{aligned} \tag{155}$$

corresponding motion equations and the fundamental Poisson bracket are

$$\dot{y}_n^a = \varepsilon_{ab}\delta_{nm}\frac{\partial H}{\partial y_m^b} = \{y_n^a, H\}, \{y_n^a, y_m^b\} = \varepsilon_{ab}\delta_{nm}. \tag{156}$$

To the canonical quantization of this system corresponds

$$[\hat{y}_n^a, \hat{y}_m^b] = i\hbar\varepsilon_{ab}\delta_{nm},\ \hat{y}_n^1 = y_n^1,\ \hat{y}_n^2 = -i\hbar\frac{\partial}{\partial y_n^1} \tag{157}$$

In this quantum theory, classical part, motion equations for y_n^1, remain classical.

Nabu – Babylonian God
of Wisdom and Writing.

The Hamiltonian mechanics (HM) is in the fundamentals of mathematical description of the physical theories [Faddeev, Takhtajan, 1990]. But HM is in a sense blind; e.g., it does not make a difference between two opposites: the ergodic Hamiltonian systems (with just one integral of motion) [Sinai, 1993] and (super)integrable Hamiltonian systems (with maximal number of the integrals of motion).
Nabu mechanics (NM) [Nambu, 1973, Whittaker, 1927] is a proper generalization of the HM, which makes the difference between dynamical systems with different numbers of integrals of motion explicit (see, e.g.[Makhaldiani, 2007]).

In the canonical formulation, the equations of motion of a physical system are defined via a Poisson bracket and a Hamiltonian, [Arnold, 1978]. In Nambu's formulation, the Poisson bracket is replaced by the Nambu bracket with $n+1, n \geq 1$, slots. For $n = 1$, we have the canonical formalism with one Hamiltonian. For $n \geq 2$, we have Nambu-Poisson formalism, with n Hamiltonians, [Nambu, 1973], [Whittaker, 1927].

The system of N vortexes can be described by the following system of differential equations, [Aref, 1983, Meleshko,Konstantinov, 1993]

$$\dot{z}_n = i \sum_{m \neq n}^{N} \frac{\gamma_m}{z_n^* - z_m^*}, \quad 1 \leq n \leq N, \tag{158}$$

where $z_n = x_n + iy_n$ are complex coordinate of the centre of n-th vortex, for $N = 3$, and the quantities

$$\begin{aligned} u_1 &= ln|z_2 - z_3|^2, \\ u_2 &= ln|z_3 - z_1|^2, \\ u_3 &= ln|z_1 - z_2|^2 \end{aligned} \tag{159}$$

reduce to the following system

$$\begin{aligned} \dot{u}_1 &= \gamma_1(e^{u_2} - e^{u_3}), \\ \dot{u}_2 &= \gamma_2(e^{u_3} - e^{u_1}), \\ \dot{u}_3 &= \gamma_3(e^{u_1} - e^{u_2}), \end{aligned} \tag{160}$$

The system (160) has two integrals of motion

$$H_1 = \sum_{i=1}^{3} \frac{e^{u_i}}{\gamma_i}, H_2 = \sum_{i=1}^{3} \frac{u_i}{\gamma_i}$$

and can be presented in the Nambu–Poisson form, [Makhaldiani, 1997,2]

$$\dot{u}_i = \omega_{ijk} \frac{\partial H_1}{\partial u_j} \frac{\partial H_2}{\partial u_k} = \{x_i, H_1, H_2\} = \omega_{ijk} \frac{e^{u_j}}{\gamma_j} \frac{1}{\gamma_k},$$

where

$$\omega_{ijk} = \epsilon_{ijk}\rho, \rho = \gamma_1\gamma_2\gamma_3$$

and the Nambu–Poisson bracket of the functions A, B, C on the three-dimensional phase space is

$$\{A, B, C\} = \omega_{ijk} \frac{\partial A}{\partial u_i} \frac{\partial B}{\partial u_j} \frac{\partial C}{\partial u_k}. \quad (161)$$

This system is superintegrable: for $N = 3$ degrees of freedom, we have maximal number of the integrals of motion $N - 1 = 2$.

As an example of the infinite dimensional Nambu-Poisson dynamics, let me conside the following extension of Schrödinger quantum mechanics [Makhaldiani, 2000]

$$iV_t = \Delta V - \frac{V^2}{2}, \tag{162}$$

$$i\psi_t = -\Delta\psi + V\psi. \tag{163}$$

An interesting solution to the equation for the potential (162) is

$$V = \frac{4(4-d)}{r^2}, \tag{164}$$

where d is the dimension of the spase. In the case of $d = 1$, we have the potential of conformal quantum mechanics.
The variational formulation of the extended quantum theory, is given by the following Lagrangian

$$L = (iV_t - \Delta V + \frac{1}{2}V^2)\psi. \tag{165}$$

The momentum variables are

$$P_v = \frac{\partial L}{\partial V_t} = i\psi, P_\psi = 0. \quad (166)$$

As Hamiltonians of the Nambu-theoretic formulation, we take the following integrals of motion

$$\begin{aligned} H_1 &= \int d^d x(\Delta V - \frac{1}{2}V^2)\psi, \\ H_2 &= \int d^d x(P_v - i\psi), \\ H_3 &= \int d^d x P_\psi. \end{aligned} \quad (167)$$

We invent unifying vector notation, $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) = (\psi, P_\psi, V, P_v)$. Then it may be verified that the equations of the extended quantum theory can be put in the following Nambu-theoretic form

$$\phi_t(x) = \{\phi(x), H_1, H_2, H_3\}, \quad (168)$$

where the bracket is defined as

$$\{A_1, A_2, A_3, A_4\} = i\varepsilon_{ijkl} \int \frac{\delta A_1}{\delta\phi_i(y)} \frac{\delta A_2}{\delta\phi_j(y)} \frac{\delta A_3}{\delta\phi_k(y)} \frac{\delta A_4}{\delta\phi_l(y)} dy$$
$$= i \int \frac{\delta(A_1, A_2, A_3, A_4)}{\delta(\phi_1(y), \phi_2(y), \phi_3(y), \phi_4(y))} dy = idet(\frac{\delta A_k}{\delta\phi_l}). \qquad (169)$$

The basic building blocks of M theory are membranes and $M5-$branes. Membranes are fundamental objects carrying electric charges with respect to the 3-form C-field, and $M5$-branes are magnetic solitons. The Nambu-Poisson 3-algebras appear as gauge symmetries of superconformal Chern-Simons nonabelian theories in 2 + 1 dimensions with the maximum allowed number of $N = 8$ linear supersymmetries.
The Bagger and Lambert [Bagger, Lambert, 2007] and, Gustavsson [Gustavsson, 2007] (BLG) model is based on a 3-algebra,

$$[T^a, T^b, T^c] = f_d^{abc} T^d \tag{170}$$

where T^a, are generators and f_{abcd} is a fully anti-symmetric tensor.

Given this algebra, a maximally supersymmetric Chern-Simons lagrangian is:

$$L = L_{CS} + L_{matter},$$
$$L_{CS} = \frac{1}{2}\varepsilon^{\mu\nu\lambda}(f_{abcd}A_\mu^{ab}\partial_\nu A_\lambda^{cd} + \frac{2}{3}f_{cdag}f_{efb}^{g}A_\mu^{ab}A_\nu^{cd}A_\lambda^{ef}), \quad (171)$$

$$L_{matter} = \frac{1}{2}B_\mu^{Ia}B_a^{\mu I} - B_\mu^{Ia}D^\mu X_a^I$$
$$+\frac{i}{2}\bar{\psi}^a\Gamma^\mu D_\mu\psi_a + \frac{i}{4}\bar{\psi}^b\Gamma_{IJ}x_c^I x_d^J\psi_a f^{abcd}$$
$$-\frac{1}{12}tr([X^I,X^J,X^K][X^I,X^J,X^K]),\ I = 1,2,...,8, \quad (172)$$

where A_μ^{ab} is gauge boson, ψ^a and $X^I = X_a^I T^a$ matter fields. If $a = 1,2,3,4$, then we can obtain an $SO(4)$ gauge symmetry by choosing $f_{abcd} = f\varepsilon_{abcd}, f$ being a constant. It turns out to be the only case that gives a gauge theory with manifest unitarity and $N = 8$ supersymmetry.

The action has the first order form so we can use the formalism of the first section. The motion equations for the gauge fields

$$f^{nm}_{abcd}\dot{A}^{cd}_m(t,x) = \frac{\delta H}{\delta A^{ab}_n(t,x)}, f^{nm}_{abcd} = \varepsilon^{nm} f_{abcd} \tag{173}$$

take canonical form

$$\dot{A}^{ab}_n = f^{abcd}_{nm}\frac{\delta H}{\delta A^{cd}_m} = \{A^{ab}_n, A^{cd}_m\}\frac{\delta H}{\delta A^{cd}_m} = \{A^{ab}_n, H\},$$
$$\{A^{ab}_n(t,x), A^{cd}_m(t,y)\} = \varepsilon_{nm} f^{abcd}\delta^{(2)}(x-y) \tag{174}$$

The quasi-classical description of the motion of a relativistic (nonradiating) point particle with spin in accelerators and storage rings includes the equations of orbit motion

$$\begin{aligned} &\dot{x}_n = f_n(x),\ f_n(x) = \varepsilon_{nm}\partial_m H,\ n, m = 1, 2, ..., 6; \\ &x_n = q_n,\ x_{n+3} = p_n,\ \varepsilon_{n,n+3} = 1,\ n = 1, 2, 3; \\ &H = e\Phi + c\sqrt{\wp^2 + m^2c^2},\ \wp_n = p_n - \frac{e}{c}A_n \end{aligned} \tag{175}$$

and Thomas-BMT equations
[Tomas, 1927, Bargmann, Michel,Telegdi, 1959] of classical spin motion

$$\begin{aligned} &\dot{s}_n = \varepsilon_{nmk}\Omega_m s_k = \{H_1, H_2, s_n\},\ H_1 = \Omega \cdot s,\ H_2 = s^2, \\ &\{A, B, C\} = \varepsilon_{nmk}\partial_n A \partial_m B \partial_k C, \end{aligned} \tag{176}$$

$$\Omega_n = \frac{-e}{m\gamma c}((1+k\gamma)B_n - k\frac{(B\cdot\wp)\wp_n}{m^2c^2(1+\gamma)} + \frac{1+k(1+\gamma)}{mc(1+\gamma)}\varepsilon_{nmk}E_m\wp_k) \tag{177}$$

where, parameters e and m are the charge and the rest mass of the particle, c is the velocity of light, $k=(g-2)/2$ quantifies the anomalous spin g factor, γ is the Lorentz factor, p_n are components of the kinetic momentum vector, E_n and B_n are the electric and magnetic fields, and A_n and Φ are the vector and scalar potentials;

$$B_n = \varepsilon_{nmk}\partial_m A_k,\ E_n = -\partial_n\Phi - \frac{1}{c}\dot{A}_n,$$
$$\gamma = \frac{H-e\Phi}{mc^2} = \sqrt{1+\frac{\wp^2}{m^2c^2}} \tag{178}$$

The spin motion equations we put in the Nambu-Poisson form. Hamiltonization of this dynamical system according to the general approach of the previous sections we will put in the ground of the optimal control theory of the accelerator.

The general method of Hamiltonization of the dynamical systems we can use also in the spinning particle case. Let us invent unified configuration space $q = (x, p, s),\ x_n = q_n,\ p_n = q_{n+3},\ s_n = q_{n+6},\ n = 1, 2, 3$; extended phase space, (q_n, ψ_n) and hamiltonian

$$H = H(q, \psi) = v_n \psi_n,\ n = 1, 2, ...9; \tag{179}$$

motion equations

$$\begin{aligned} \dot{q}_n &= v_n(q), \\ \dot{\psi}_n &= -\frac{\partial v_m}{\partial q_n}\psi_m \end{aligned} \tag{180}$$

where the velocities v_n depends on external fields as in previous section as control parameters which can be determined according to the optimal control criterium.

EDM are one of the keys to understand the origin of our Universe [Sakharov, 1967]. Andrei Sakharov formulated three conditions for baryogenesis:
1. Early in the evolution of the universe, the baryon number conservation must be violated sufficiently strongly,
2. The C and CP invariances, and T invariance thereof, must be violated, and
3. At the moment when the baryon number is generated, the evolution of the universe must be out of thermal equilibrium.
CP violation in kaon decays is known since 1964, it has been observed in B-decays and charmed meson decays. The Standard Model (SM) accommodates CP violation via the phase in the Cabibbo-Kobayashi-Maskawa matrix.
CP and P violation entail nonvanishing P and T violating electric dipole moments (EDM) of elementary particles $\vec{d} = d\vec{s}$.

Although extremely successful in many aspects, the SM has at least two weaknesses: neutrino oscillations do require extensions of the SM and, most importantly, the SM mechanisms fail miserably in the expected baryogenesis rate.
Simultaneously, the SM predicts an exceedingly small electric dipole moment of nucleons $10^{-33} < d_n < 10^{-31} e \cdot cm$, way below the current upper bound for the neutron EDM, $d_n < 2.9 \times 10^{-26} e \cdot cm$. In the quest for physics beyond the SM one could follow either the high energy trail or look into new methods which offer very high precision and sensitivity.
Supersymmetry is one of the most attractive extensions of the SM and S. Weinberg emphasized [Weinberg, 1993]: "Endemic in supersymmetric (SUSY) theories are CP violations that go beyond the SM. For this reason it may be that the next exciting thing to come along will be the discovery of a neutron electric dipole moment."

The SUSY predictions span typically $10^{-29} < d_n < 10^{-24} e \cdot cm$ and precisely this range is targeted in the new generation of EDM searches [Roberts, Marciano, 2010]. There is consensus among theorists that measuring the EDM of the proton, deuteron and helion is as important as that of the neutron. Furthermore, it has been argued that T-violating nuclear forces could substantially enhance nuclear EDM [Flambaum, Khriplovich, Sushkov, 1986]. At the moment, there are no significant direct upper bounds available on d_p or d_d. Non-vanishing EDMs give rise to the precession of the spin of a particle in an electric field. In the rest frame of a particle

$$\dot{s}_n = \varepsilon_{nmk}(\Omega_m s_k + d_m E_k),\ \Omega_m = -\mu B_m, \tag{181}$$

where in terms of the lab frame fields

$$\begin{aligned} B_n &= \gamma(B_n^l - \varepsilon_{nmk}\beta_m E_k^l), \\ E_n &= \gamma(E_n^l + \varepsilon_{nmk}\beta_m B_k^l) \end{aligned} \tag{182}$$

Now we can apply the Hamiltonization and optimal control theory methods to this dynamical system.

Note that the procedure of reduction of the higher order dynamical system, e.g. second order Euler-Lagrange motion equations, to the first order dynamical systems, in the case to the Hamiltonian motion equations, can be continued using fractal calculus. E.g. first order system can be reduced to the half order one,

$$\begin{aligned} &D^{1/2}q = \psi, \\ &D^{1/2}\psi = p \Leftrightarrow \dot{q} = p. \end{aligned} \tag{183}$$

Computers are physical devices and their behavior is determined by physical laws. The Quantum Computations [Benenti, Casati, Strini, 2004 , Nielsen, Chuang, 2000], Quantum Computing, Quanputing [Makhaldiani, 2007.2], is a new interdisciplinary field of research, which benefits from the contributions of physicists, computer scientists, mathematicians, chemists and engineers. Contemporary digital computer and its logical elements can be considered as a spatial type of discrete dynamical systems [Makhaldiani, 2001]

$$S_n(k+1) = \Phi_n(S(k)), \tag{184}$$

where

$$S_n(k), \quad 1 \leq n \leq N(k), \tag{185}$$

is the state vector of the system at the discrete time step k. Vector S may describe the state and Φ transition rule of some Cellular Automata [Toffoli, Margolus, 1987].The systems of the type (184) appears in applied mathematics as an explicit finite difference scheme approximation of the equations of the physics [Samarskii, Gulin, 1989].

Definition: *We assume that the system (184) is time-reversible if we can define the reverse dynamical system*

$$S_n(k) = \Phi_n^{-1}(S(k+1)). \tag{186}$$

In this case the following matrix

$$M_{nm} = \frac{\partial \Phi_n(S(k))}{\partial S_m(k)}, \tag{187}$$

is regular, i.e. has an inverse. If the matrix is not regular, this is the case, for example, when $N(k+1) \neq N(k)$, we have an irreversible dynamical system (usual digital computers and/or corresponding irreversible gates).

Let us consider an extension of the dynamical system (184) given by the following action function

$$A = \sum_{kn} l_n(k)(S_n(k+1) - \Phi_n(S(k))) \tag{188}$$

and corresponding motion equations

$$\begin{aligned} S_n(k+1) &= \Phi_n(S(k)) = \frac{\partial H}{\partial l_n(k)}, \\ l_n(k-1) &= l_m(k)\frac{\partial \Phi_m(S(k))}{\partial S_n(k)} = l_m(k)M_{mn}(S(k)) = \frac{\partial H}{\partial S_n(k)}, \end{aligned} \tag{189}$$

where

$$H = \sum_{kn} l_n(k)\Phi_n(S(k)), \tag{190}$$

is discrete Hamiltonian. In the regular case, we put the system (189) in an explicit form

$$\begin{aligned} S_n(k+1) &= \Phi_n(S(k)), \\ l_n(k+1) &= l_m(k)M^{-1}_{mn}(S(k+1)). \end{aligned} \tag{191}$$

From this system it is obvious that, when the initial value $l_n(k_0)$ is given, the evolution of the vector $l(k)$ is defined by evolution of the state vector $S(k)$. The equation of motion for $l_n(k)$ - Elenka is linear and has an important property that a linear superpositions of the solutions are also solutions.

Statement: *Any time-reversible dynamical system (e.g. a time-reversible computer) can be extended by corresponding linear dynamical system (quantum - like processor) which is controlled by the dynamical system and has a huge computational power,*
[Makhaldiani, 2001, Makhaldiani, 2002, Makhaldiani, 2007.2, Makhaldiani, 2011.2].

For motion equations (189) in the continual approximation, we have

$$\begin{aligned} &S_n(k+1) = x_n(t_k+\tau) = x_n(t_k) + \dot{x}_n(t_k)\tau + O(\tau^2), \\ &\dot{x}_n(t_k) = v_n(x(t_k)) + O(\tau), \quad t_k = k\tau, \\ &v_n(x(t_k)) = (\Phi_n(x(t_k)) - x_n(t_k))/\tau; \\ &M_{mn}(x(t_k)) = \delta_{mn} + \tau \frac{\partial v_m(x(t_k))}{\partial x_n(t_k)}. \end{aligned} \tag{192}$$

(de)Coherence criterion: *the system is reversible, the linear (quantum, coherent, soul) subsystem exists, when the matrix M is regular,*

$$detM = 1 + \tau \sum_n \frac{\partial v_n}{\partial x_n} + O(\tau^2) \neq 0. \tag{193}$$

For the Nambu - Poisson dynamical systems (see e.g. [Makhaldiani, 2007])

$$\begin{aligned} &v_n(x) = \varepsilon_{nm_1m_2...m_p} \frac{\partial H_1}{\partial x_{m_1}} \frac{\partial H_2}{\partial x_{m_2}} ... \frac{\partial H_p}{\partial x_{m_p}}, \quad p = 1, 2, 3, ..., N-1, \\ &\sum_n \frac{\partial v_n}{\partial x_n} \equiv divv = 0. \end{aligned} \tag{194}$$

Let me motivate an idea of construction of the reversible dynamical systems by simple example from field theory. There are renormalizable models of scalar field theory of the form (see, e.g. [Makhaldiani, 1980])

$$L = \frac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2) - g\varphi^n, \tag{195}$$

with the constraint

$$n = \frac{2d}{d-2}\ , \tag{196}$$

where d is dimension of the space-time and n is degree of nonlinearity. It is interesting that if we define d as a function of n, we find

$$d = \frac{2n}{n-2} \tag{197}$$

the same function !

Thing is that, the constraint can be put in the symmetric implicit form [Makhaldiani, 1980]

$$\frac{1}{n} + \frac{1}{d} = \frac{1}{2} \tag{198}$$

Now it is natural to consider the following symmetric function

$$f(y) + f(x) = c \tag{199}$$

and define its solution

$$y = f^{-1}(c - f(x)). \tag{200}$$

This is the general method, that we will use in the following construction of the reversible dynamical systems. In the simplest case,

$$f(x) = x, \tag{201}$$

we take

$$y = S(k+1), \quad x = S(k-1), \quad c = \tilde{\Phi}(S(k)) \tag{202}$$

and define our reversible dynamical system from the following symmetric, implicit form (see also [Toffoli, Margolus, 1987])

$$S(k+1) + S(k-1) = \tilde{\Phi}(S(k)), \tag{203}$$

explicit form of which is

$$\begin{aligned} S(k+1) &= \Phi(S(k), S(k-1)) \\ &= \tilde{\Phi}(S(k)) - S(k-1). \end{aligned} \tag{204}$$

This dynamical system defines given state vector by previous two state vectors. We have reversible dynamical system on the time lattice with time steps of two units,

$$
\begin{aligned}
&S(k+2,2) = \Phi(S(k,2)),\\
&S(k+2,2) \equiv (S(k+2), S(k+1)),\\
&S(k,2) \equiv (S(k), S(k-1))).
\end{aligned}
\tag{205}
$$

Starting from a general discrete dynamical system, we obtained reversible dynamical system with internal(spin,bit) degrees of freedom

$$\begin{aligned} S_{ns}(k+2) &\equiv \begin{pmatrix} S_n(k+2) \\ S_n(k+1) \end{pmatrix} = \begin{pmatrix} \Phi_n(\Phi(S(k)) - S(k-1)) - S(k)) \\ \Phi_n(S(k)) - S_n(k-1) \end{pmatrix} \\ &\equiv \Phi_{ns}(S(k)), \quad s = 1, 2 \end{aligned} \tag{206}$$

where

$$S(k) \equiv (S_{ns}(k)), \quad S_{n1}(k) \equiv S_n(k), \quad S_{n2}(k) \equiv S_n(k-1) \tag{207}$$

For the extended system we have the following action

$$A = \sum_{kns} l_{ns}(k)(S_{ns}(k+2) - \Phi_{ns}(S(k))) \tag{208}$$

and corresponding motion equations

$$
\begin{aligned}
S_{ns}(k+2) &= \Phi_{ns}(S(k)) = \frac{\partial H}{\partial l_{ns}(k)}, \\
l_{ns}(k-2) &= l_{mt}(k)\frac{\partial \Phi_{mt}(S(k))}{\partial S_{ns}(k)} \\
&= l_{mt}(k) M_{mtns}(S(k)) = \frac{\partial H}{\partial S_{ns}(k)},
\end{aligned}
\tag{209}
$$

By construction, we have the following reversible dynamical system

$$
\begin{aligned}
S_{ns}(k+2) &= \Phi_{ns}(S(k)), \\
l_{ns}(k+2) &= l_{mt}(k) M^{-1}_{mtns}(S(k+2)),
\end{aligned}
\tag{210}
$$

with classical S_{ns} and quantum l_{ns}(in the external, background S) string bit dynamics.

We can also consider p-point generalization of the previous structure,

$$\begin{aligned}
&f_p(S(k+p)) + f_{p-1}(S(k+p-1)) + ... + f_1(S(k+1)) \\
&+f_1(S(k-1)) + ... + f_p(S(k-p)) = \tilde{\Phi}(S(k)), \\
&S(k+p) = \Phi(S(k), S(k+p-1), ..., S(k-p)) \\
&\equiv f_p^{-1}(\tilde{\Phi}(S(k)) - f_{p-1}(S(k+p-1)) - ... - f_p(S(k-p)))
\end{aligned} \tag{211}$$

and corresponding reversible p-oint cluster dynamical system

$$\begin{aligned}
&S(k+p,p) \equiv \Phi(S(k,p)), \\
&S(k+p,p) \equiv (S(k+p), S(k+p-1), ..., S(k+1)), \\
&S(k,p) \equiv (S(k), S(k-1), ..., S(k-p+1)), \quad S(k,1) = S(k).
\end{aligned} \tag{212}$$

So we have general method of construction of the reversible dynamical systems on the time (tame) scale p. The method of linear extension of the reversible dynamical systems (see [Makhaldiani, 2001] and previous section) defines corresponding Quanputers,

$$\begin{aligned}
&S_{ns}(k+p) = \Phi_{ns}(S(k)), \\
&l_{ns}(k+p) = l_{mt}(k) M^{-1}_{mtns}(S(k+p)),
\end{aligned} \tag{213}$$

p-point cluster and higher spin states reversible dynamics, or pit string dynamics

This case the quantum state function $l_{ns},\ s = 1, 2, ...p$ will describes the state with spin $(p-1)/2$.

Note that, in this formalism for reversible dynamics minimal value of the spin is $1/2$. There is not a place for a scalar dynamics, or the scalar dynamics is not reversible. In the Standard model (SM) of particle physics, [Beringer et al, 2012], all of the fundamental particles, leptons, quarks and gauge bosons have spin. Only scalar particles of the SM are the Higgs bosons. Perhaps the scalar particles are composed systems or quasiparticles like phonon, or Higgs dynamics is not reversible (a mechanism for 'time arrow').

A way to the Solution of the Traveling salesman problem (TSP) with Quanputing

The $NP \stackrel{?}{=} P$ problem will be solved if for some $NP-$ complete problem, e.g. TSP, a polynomial algorithm find; or show that there is not such an algorithm; or show that it is impossible to find definite answer to that question.
TSP means to find minimal length path between N fixed points on a surface, which attends any point ones. We consider a system where N points with quenched positions $x_1, x_2, ..., x_N$ are independently distributed on a finite domain D with a probability density function $p(x)$. In general, the domain D is multidimensional and the points x_n are vectors in the corresponding Euclidean space. Inside the domain D we consider a polymer chain composed of N monomers whose positions are denoted by $y_1, y_2, ..., y_N$. Each monomer y_n is attached to one of the quenched sites x_m and only one monomer can be attached to each site. The state of the polymer is described by a permutation $\sigma \in \Sigma_N$ where Σ_N is the group of permutations of N objecs.

A way to the Solution of the Traveling salesman problem (TSP) with Quanputing

The Hamiltonian for the system is given by

$$H = \sum_{n=1}^{N} V(|y_n - y_{n-1}|) \tag{214}$$

Here V is the interaction between neighboring monomers on the polymer chain. For convenience the chain is taken to be closed, thus we take the periodic boundary condition $x_0 = x_N$. A physical realization of this system is one where the x_n are impurities where the monomers of a polymer loop are pinned. In combinatorial optimization, if one takes $V(x)$ to be the norm, or distance, of the vector x then $H(\sigma)$ is the total distance covered by a path which visits each site x_n exactly once. The problem of finding σ_0 which minimizes $H(\sigma)$ is known as the traveling salesman problem (TSP) [Gutin, Pannen, 2002].

In field theory language to the TSP we correspond the calculation of the following correlator

$$
\begin{aligned}
&G_{2N}(x_1,x_2,...,x_N) = Z_0^{-1}\int d\varphi(x)\varphi^2(x_1)\varphi^2(x_2)...\varphi^2(x_N)e^{-S(\varphi)}\\
&= \frac{\delta^{2N}F(J)}{\delta J(x_1)^2...\delta J(x_N)^2},\ F(J)=\ln Z(J),\\
&Z(J)=\int d\varphi e^{-\frac{1}{2}\varphi\cdot A\cdot\varphi+J\cdot\varphi} = e^{\frac{1}{2}J\cdot A^{-1}\cdot J},\ A^{-1}(x,y;m)=e^{-m|x-y|},\\
&L_{min}(x_1,...,x_N) = -\frac{d}{dm}\ln G_{2Ns}+O(e^{-am})\\
&< A^{-1} > \equiv \frac{1}{\Gamma(s)}\int_0^\infty dmm^{s-1}A^{-1}(x,y;m) = \frac{1}{|x-y|^s}\\
&= L_sA^{-1}(x-y;s)\\
&k(d)\Delta_dL_sA^{-1}(x;s)=\delta^d(x)\Rightarrow A(x;s)=k(d)\Delta_dL_s,\\
&s=d-2;\varphi=\varphi(x,m).
\end{aligned}
\tag{215}
$$

If we take relativistic massive scalar field, then $A = \Delta_d + m^2$,

$$A^{-1}(x) \sim |x|^{2-d} e^{-m|x|}, \tag{216}$$

and for $d = 2$, we also have the needed behaviour. Note that G_{2N} is symmetric with respect to its arguments and contains any paths including minimal length one.

K. Aamodt et al. [ALICE collaboration] *Eur. Phys. J.* **C65** (2010) 111 [arXiv:1004.3034], [arXiv:1004.3514].

M. Anselmino, B.L. Ioffe and E. Leader, Sov. J. Nucl. 49, 136, (1989)

H. Aref, Ann. Rev. Fluid Mech. **15** (1983) 345.

V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, New York 1978.

J. Bagger, N. Lambert, *Modeling multiple M2's*, Phys. Rev. D **75** (2007) 045020; [arXiv:hep-th/0611108].

V.V. Balandin, N.I. Golubeva, *Hamiltonian Methods for the Study of Polarized Proton Beam Dynamics in Accelerators and Storage Rings*, [arXiv://arxiv.org/abs/physics/9903032v1].

V. Bargmann, L. Michel, V.L. Telegdi, *Precession of the polarization of particles moving in a homogeneous electromagnetic field*, Phys. Rev. Lett., **2(10)** (1959) 435.

G. Benenti, G. Casati, G. Strini, *Principles of quantum computation and information*, Vol. I: Basic concepts, World Scientific, Singapore 2004; Vol. II: *Basic tools and special topics* World Scientific, Singapore 2007.

J. Beringer et al. (Particle Data Group) *Review of Particle Physics*, Phys. Rev. D **86** (2012) 010001 [1528 pages]

J. D. Bjorken, Phys. Rev. 148, 1467 (1966).

F.A. Berezin, *Introduction to Superanalysis*, Reidel, Dordrecht 1987.

F.A. Berezin, M.S. Marinov, Ann. Phys. (N.Y.) **104** (1977) 336.

N.N. Bogoliubov and D.V. Shirkov, *Introduction to the Theory of Quantized Fields*, New York 1959.

S. Brodsky, G. Farrar, Phys. Rev. Lett. **31** 1153 (1973).

Stanley J. Brodsky, Guy F. de Tèramond, Alexandre Deur, Nonperturbative QCD Coupling and its β function from Light-Front Holography, PRD 81,096010 (2010), arXiv:1002.3948

V. D. Burkert, Phys. Rev. D 63, 097904 (2001).

C. Buttin, C.R. Acad. Sci. Paris. **269** (1969) 87.

W. E. Caswell, Phys. Rev. Lett. 33 (1974) 244.

J.C. Collins, *Renormalization*, Cambrige Univ. Press, London 1984.

Ya.Z. Darbaidze, N.V. Makhaldiani, A.N. Sisakian, L.A. Slepchenko, *TMF* **34** (1978) 303.

D. Diakonov, Prog. Par. Nucl. Phys. 51 (2003) 173.

P.A.M. Dirac, *Proc. Roy, Soc.* **A167** (1938) 148.

S. D. Drell, A. C. Hearn, *Exact Sum Rule For Nucleon Magnetic Moments*, *Phys. Rev. Lett.* **16** (1966) 908.

E. Egorian and O. V. Tarasov, Theor. Math. Phys. 41 (1979) 863 [Teor. Mat. Fiz. 41 (1979) 26]

W. Ernst, I. Schmitt, *Nuovo Cim.* **A33** (1976) 195.

L.D. Faddeev, R. Jackiw, Phys.Rev.Lett. **60** (1988) 1692.

L.D. Faddeev and L.A. Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer, Berlin 1990.

C. S. Fischer and H. Gies, JHEP 0410 (2004) 048.

V. Flambaum, I. Khriplovich and O. Sushkov, Nucl.Phys. A449, p. 750 (1986).

S.B. Gerasimov, *A Sum Rule for Magnetic Moments and Damping of the Nucleon Magnetic Moment in Nuclei*, *J.Nucl.Phys.(USSR)* **2** (1965) 598.

V. L. Ginzburg, *Theoretical Physics and Astrophysics*, Pergamon, New York 1979.

M.L. Goldberger, S.B. Treiman, Phys. Rev. **110** 1178 (1958).

D.J. Gross, F. Wilczek, Phys. Rev. Lett. **30** 1343 (1973).

S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109];

A. Gustavsson, *Algebraic structures on parallel M2-branes*, Nucl. Phys. B.**811** (2009) 66; [arXiv:0709.1260 [hep-th]].

G. Gutin, A.P. Punnen (Eds), *The Traveling Salesman Problem and Its Variations*, Combinatorial Optimization Series, Kluwer, Boston 2002.

W. Heisenberg, *Introduction to the Unified field Theory of Elementary particles,* Interscience Publishers, London 1966.

G. 't Hooft, report at the *Marseille Conference on Yang-Mills Fields*, 1972.

G. 't Hooft, *Nucl.Phys.* **B 61** (1973) 455.

J.D. Jackson, *Classical Electrodynamics, 3rd ed.* JohnWiley & Sons, Inc. New York 1999.

X. Ji, J.Osborne, J.Phys. G27 127 (2001).

D. R. T. Jones, Nucl. Phys. B 75 (1974) 531;

M. Kaku, *Strings, Conformal Fields, and M-Theory*, Springer, New York 2000.

D.I. Kazakov, *Supersymmetric Generalization of the Standard Model of Fundamental Interactions*, Textbook, JINR Dubna 2004.

D.I. Kazakov, D.V. Shirkov, Fortschr. d. Phys. **28** 447 (1980).

D.I. Kazakov, L.R. Lomidze, N.V. Makhaldiani, A.A. Vladimirov, *Ultraviolet Asymptotics in Renormalizable Scalar Theories*, JINR Communications, **E2-8085**, Dubna 1974.

A.Yu. Kitaev, A. Shen, M.N. Vyalyi, *Classical and Quantum Computation*, American Mathematical Society, 2002.

Z. Koba, H.B. Nielsen, P. Olesen, *Nucl. Phys.* **B40** (1972) 317.

N.V. Makhaldiani, *Approximate methods of the field theory and their applications in physics of high energy, condensed matter, plasma and hydrodynamics*, Dubna 1980.

S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B 303 (1993) 334.

C. Lerche and L. von Smekal, Phys. Rev. D 65 (2002) 125006.

N.M. Makhaldiani, *Computational Quantum Field Theory*, JINR Communication, **P2-86-849**, Dubna 1986.

N.V. Makhaldiani, *A New Approach to the Problem of Space Compactification*, JINR Communications, **P2-87-306**, Dubna 1987.

N.V. Makhaldiani, *Number Fields Dynamics and the Compactification of Space Problem in the Unified Theories of Fields and Strings*, JINR Communications, **P2-88-916**, Dubna 1988.

N. Makhaldiani, *The System of Three Vortexes of Two-Dimensional Ideal Hydrodynamics as a New Example of the (Integrable) Nambu-Poisson Mechanics*, JINR Communications **E2-97-407**, Dubna 1997; [arXiv:solv-int/9804002].

N. Makhaldiani, *The Algebras of the Integrals of Motion and Modified Bochner-Killing-Yano Structures of the Point particle Dynamics*, JINR Communications **E2-99-337**, Dubna 1999.

N. Makhaldiani, *New Hamiltonization of the Schrödinger Equation by Corresponding Nonlinear Equation for the Potential*, JINR Communications, **E2-2000-179**, Dubna 2000.

N. Makhaldiani, *How to Solve the Classical Problems on Quantum Computers*, JINR Communications, **E2-2001-137**, Dubna 2001.

N. Makhaldiani, *Classical and Quantum Problems for Quanputers*, [arXiv:quant-ph/0210184].

N. Makhaldiani, *Adelic Universe and Cosmological Constant*, JINR Communications, **E2-2003-215** Dubna 2003, [arXiv:hep-th/0312291].

N. Makhaldiani, *Nambu-Poisson dynamics of superintegrable systems*, *Atomic Nuclei*, **70** (2007) 564.

N. Makhaldiani, *Theory of Quanputers*, Sovremennaia Matematica i ee Prilozhenia, **44** (2007) 113; Journal of Mathematical Sciences, **153** (2008) 159.

N.V. Makhaldiani, *Renormdynamics and Scaling Functions*, in Proc. of the *XIX International Baldin Seminar on High Energy Physics Problems* eds. A.N.Sissakian, V.V.Burov, A.I.Malakhov, S.G.Bondartenko, E.B.Plekhanov, Vol.II, p. 175, Dubna 2008.

N.V. Makhaldiani, *Renormdynamics, multiparticle production, negative binomial distribution and Riemann zeta function*, [arXiv:1012.5939v1 [math-ph]] 24 Dec 2010.

N. Makhaldiani, *Fractal Calculus (H) and some Applications, Physics of Particles and Nuclei Letters*, **8** 325 (2011).

N. Makhaldiani, *Regular method of construction of the reversible dynamical systems and their linear extensions - Quanputers*, Atomic Nuclei, **74** (2011) 1040.

Nugzar Makhaldiani, *Nambu-Poisson Dynamics with Some Applications*,
Physics of Particles and Nuclei, **43** (2012) 703.

Nugzar Makhaldiani, *Renormdynamics, coupling constant unification and universal properties of the multiparticle production*, XXI International Baldin Seminar on High Energy Physics Problems, September 10-15, 2012, PoS(Baldin ISHEPP XXI)068.

N. V. Makhaldiani, *Renormdynamics, Multiparticle Production, Negative Binomial Distribution, and Riemann Zeta Function*, Physics of Atomic Nuclei, **76** 1169 (2013).

N. Makhaldiani, O. Voskresenskaya, *On the correspondence between the dynamics with odd and even brackets and generalized Numbu's mechanics,*
JINR Communications, **E2-97-418**, Dubna 1997.

J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

V. Matveev, R. Muradyan, A. Tavkhelidze, Lett. Nuovo Cimento **7** 719 (1973).

V.A. Matveev, A.N. Sisakian, L.A. Slepchenko, *Nucl. Phys.* **23** (1976) 432.

A.V. Meleshko, N.N. Konstantinov, *Dynamics of vortex systems*,
Naukova Dumka, Kiev 1993.

Y. Nambu, Phys.Rev. D **7** (1973) 2405.

M.A. Nielsen, I.L. Chuang, *Quantum computation and quantum information*, Cambridge University Press, Cambridge 2000.

J. M. Pawlowski, D. F. Litim, S. Nedelko and L. von Smekal, Phys. Rev. Lett. 93 (2004) 152002.

H.D. Politzer, Phys. Rev. Lett. **30** 1346 (1973).

M.C.M. Rentmeester, R.G.E. Timmermans, J.L. Friar, J.J. de Swart, Phys. Rev. Lett. **82** 4992 (1999).

T. van Ritbergen, J. A. M. Vermaseren and S. A. Larin, Phys. Lett. B 400 (1997)379.

B. L. Roberts and W. J. Marciano (eds.), Lepton Dipole Moments, Advanced Series on Directions in High Energy Physics, Vol. 20 (World Scientific, 2010).

A. Sakharov, Pisma Zh.Eksp.Teor.Fiz. 5, 32 (1967), Reprinted in Sov. Phys. Usp. 34 (1991) 392-393 [Usp. Fiz. Nauk 161 (1991) No. 5 61-64].

A. Samarskii, A. Gulin, *Numerical Methods*, Nauka, Moscow 1989.

Ya.G. Sinai, *Topics in Ergodic Theory*, Princeton University Press, Princeton NJ 1993.

O. V. Tarasov, A. A. Vladimirov and A. Y. Zharkov, Phys. Lett. B 93 (1980) 429.

Norman Margolus, Tommaso Toffoli, *Cellular Automaton Machines*,
MIT Press, Cambridge 1987.

M.V. Tokarev, I. Zborovský, *Z-Scaling in the Proton-Proton Collisions at RHIC*, in *Investigations of Properties of Nuclear Matter at High Temperature and Densities, Editid by A.N. Sisakian, F.A. Soifer*, Dubna 2007

L.H. Thomas, Philos. Mag. **3** (1927) 1.

M.B. Voloshin, K.A. Ter-Martyrosian, *Gauge Theory of Elementary Particles*, Atomizdat, Moscow 1984.

S. Weinberg, AIP Conf.Proc. 272, 346 (1993).

E.T. Whittaker, *A Treatise on the Analytical Dynamics*, Cambridge 1927.

E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

D. Zwanziger, Phys. Rev. D 65 (2002) 094039.