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Parton distributions at low x and gluon- and quark average multiplicities

OUTLINE

1. Introduction
2. Results.
3. Conclusions and Prospects

GENERAL

The talk has two parts:

1. A. Parton distributions at low x
2. B. gluon- and quark average multiplicities.

The main common property of both parts is diagonalization.

After diagonalization in both the parts there are two components: "$+$" one and "$-$" one.

The "$+$" component contains singularities at $N \to 1$ in the corresponding anomalous dimensions and coefficient functions (N is Mellin moment number) and requires some resummations. Usually the component is the object of study.

The “$-$” component is free of singularities at $N \to 1$ in the corresponding anomalous dimensions and coefficient functions. It has very slow Q^2-dependence (in both the considered cases).
!!! But it is strongly necessary for the agreement with experimental data. !!!

A1. Introduction to DIS

A. Deep-inelastic scattering cross-section:

$$\sigma \sim L^{\mu\nu} F^{\mu\nu}$$

Hadron part $F^{\mu\nu}$ ($Q^2 = -q^2 > 0$, $x = Q^2/[2(pq)]$):

$$\begin{aligned} F^{\mu\nu} &= (-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}) F_1(x, Q^2) \\ &- (p^\mu - \frac{(pq)}{q^2} q^\mu)(p^\nu - \frac{(pq)}{q^2} q^\nu) \frac{2x}{q^2} F_2(x, Q^2) + ..., \end{aligned}$$

where $F_k(x, Q^2)$ ($k = 1, 2, 3, L$) - are DIS SF and q and p are photon and hadron (parton) momentums.

B. Wilson operator expansion: Mellin moments $M_k(n, Q^2)$ of DIS SF $F_k(x, Q^2)$ can be represented as sum

$$M_k(n, Q^2) = \sum_{a=NS,SI,g} \underbrace{C_k^a(n, Q^2/\mu^2)}_{\text{Coeff. function}} A_a(n, \mu^2),$$

where $A_a(n, \mu^2) =< N|\mathcal{O}^a_{\mu_1,\ldots,\mu_n}|N >$ are matrix elements of the Wilson operators $\mathcal{O}^a_{\mu_1,\ldots,\mu_n}$.

C. The matrix elements $A_a(n, \mu^2)$ are Mellin moments of the unpolarized PD $f_a(n, \mu^2)$.

DGLAP [= Renormgroup] equations:

$$\frac{d}{d\ln Q^2} f_a(x, Q^2) = \int_x^1 \frac{dy}{y} \sum_b W_{b\to a}(x/y)\, f_b(y, Q^2)\,. \qquad (1)$$

The anomalous dimensions (AD) $\gamma_{ab}(n)$ of the twist-2 Wilson operators $\mathcal{O}^a_{\mu_1,\ldots,\mu_n}$ (hereafter $a_s = \alpha_s/(4\pi)$)

$$\gamma_{ab}(n) = \int_0^1 dx\; x^{n-1} W_{b\to a}(x) = \sum_{m=0}^{\infty} \gamma_{ab}^{(m)}(n) a_s^m,$$

All parton densities are multiplies by x, t.e.

structure function = combination of parton densities.

A2. Method

(C.Lopez and F.J.Yndurain, 1980,1981), (A.V.K., 1994)

Here I present briefly the method, which leads to the possibility to replace the Mellin convolution of two functions

$$f_1(x) \otimes f_2(x) \equiv \int_x^1 \frac{dy}{y} f_1(y) f_2(x/y)$$

by a simple products at small x.

So, if $f_1(x) = B_k(x,Q^2)$ is perturbatively calculated Wilson kernel and $f_2(x) = xf_a(x,Q^2) \sim x^{-\delta}$ at $x \to 0$, then

$$f_1(x) \otimes f_2(x) \;\approx\; M_k(1+\delta,Q^2)\, f_2(x) \qquad (2)$$

where $M_k(1+\delta,Q^2)$ is the analytical continuation to non-integer arguments of the Mellin moment $M_k(n,Q^2)$ of $B_k(x,Q^2)$:

$$M_k(n,Q^2) \;=\; \int_0^1 x^{n-2} B_k(x,Q^2) \qquad (3)$$

The equation (2) is correct if the moment $M_k(n,Q^2)$ has no singularity at $n \to 1$.

Remember !!! The Mellin convolution of two functions

$$f_1(x) \otimes f_2(x) \;\equiv\; \int_x^1 \frac{dy}{y} f_1(y) f_2(x/y)$$

A3. Generalized double-logarithmic approach

(A.V.K. and G.Parente, 1998),
(A.Yu.Illarionov, A.V.K. and G.Parente, 2004)

(Generalized double asymptotic scaling)

1 Leading order without quarks (a pedagogical example)

At the momentum space, the solution of the DGLAP equation is

$$M_g(n,Q^2) \; = \; M_g(n,Q_0^2)e^{-d_{gg}(n)s},$$

where $M_g(n,Q^2)$ are the moments of the gluon distribution,

$$s = ln\left(\frac{a_s(Q_0^2)}{a_s(Q^2)}\right), \quad a_s(Q^2) = \frac{\alpha_s(Q^2)}{4\pi} \quad \text{and} \quad d_{gg} = \frac{\gamma_{gg}^{(0)}(n)}{2\beta_0}$$

The terms $\gamma_{gg}^{(0)}(n)$ and β_0 are respectively the LO coefficients of the gluon-gluon AD and the QCD β-function.

For any perturbatively calculable variable $Q(n)$, it is very convenient to separate the singular part when $n \to 1$ (denoted by "$\widehat{Q}$") and the regular part (marked as "$\overline{Q}$"):

$$Q(n) \; = \; \frac{\widehat{Q}}{n-1} + \overline{Q}(n)$$

Then, the above equation can be represented by the form

$$M_g(n,Q^2) \; = \; M_g(n,Q_0^2) e^{-\hat{d}_{gg} s_{LO}/(n-1)} e^{-\overline{d}_{gg}(n) s_{LO}},$$

with $\hat{\gamma}_{gg} = -8C_A$ and $C_A = N$ for $SU(N)$ group.

Finally, if one takes the flat boundary conditions
(i.e. min information about initial conditions, or
min contribution from initial conditions.)

$$xf_a(x,Q_0^2) \; = \; A_a, \quad \to \quad M_a(n,Q_0^2) \; = \; \frac{A_a}{n-1} \qquad (4)$$

(A.D.Rujula, S.L.Glashow, H.D.Politzer, S.B.Treiman, F.Wilczek and A.Zee, 1974)

Then, expanding the second exponential in the above equation

$$M_g^{cdl}(n, Q^2) = A_g \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\left(-\hat{d}_{gg} s_{LO}\right)^k}{(n-1)^{k+1}}$$

and using the Mellin transformation for $(ln(1/x))^k$:

$$\int_0^1 dx x^{n-2} (ln(1/x))^k = \frac{k!}{(n-1)^{k+1}}$$

we immediately obtain the well known double-logarithmic behavior

$$f_g^{cdl}(x, Q^2) = A_g \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(-\hat{d}_{gg} s_{LO}\right)^k (ln(1/x))^k = A_g I_0(\sigma_{LO}),$$

where $I_0(\sigma_{LO})$ is the modified Bessel function with argument $\sigma_{LO} = 2\sqrt{\hat{d}_{gg} s_{LO} ln(x)}$. (R.D.Ball and S.Forte, 1994),

1.2 The more general case

For a regular kernel $\tilde{K}(x)$, having Mellin moment (nonsingular at $n \to 1$)

$$K(n) \; = \; \int_0^1 dx x^{n-2} \tilde{K}(x)$$

and the PD $f_a(x)$ in the form $I_\nu(\sqrt{\hat{d} ln(1/x)})$ we have the following equation

$$\tilde{K}(x) \otimes f_a(x) \; = \; K(1) f_a(x) + O(\sqrt{\frac{\hat{d}}{ln(1/x)}})$$

So, one can find the general solution for the LO gluon density without the influence of quarks

$$f_g(x,Q^2) \; = \; A_g I_0(\sigma_{LO}) e^{-\overline{d}_{gg}(1)s_{LO}} \; + \; O(\rho_{LO}),$$

where (R.D.Ball and S.Forte, 1994)

$$\rho_{LO} = \sqrt{\frac{\hat{d}_{gg}s_{LO}}{ln(x)}} = \frac{\sigma_{LO}}{2ln(1/x)}\,, \quad \overline{\gamma}^{(0)}_{gg}(1) = 22 + \frac{4}{3}f$$

and

$$\overline{d}_{gg}(1) = 1 + \frac{4f}{3\beta_0}$$

with f as the number of active quarks.

At the momentum space, the solution of the DGLAP equation at LO has the form (*after diagonalization*)

$M_a(n,Q^2) \; = \; M_a^+(n,Q^2) + M_a^-(n,Q^2)$ and

$M_a^\pm(n,Q^2) \; = \; M_a^\pm(n,Q_0^2)e^{-d_\pm(n)s} = M_a^\pm e^{-\hat{d}_\pm s/(n-1)} e^{-\overline{d}_\pm(n)s}$,

where ($\varepsilon_{ab}^\pm(n)$ are projectors)

$$M_a^\pm(n,Q^2) \; = \; \varepsilon_{ab}^\pm(n) M_b(n,Q^2), \qquad d_{ab} = \frac{\gamma_{ab}^{(0)}(n)}{2\beta_0}, \tag{5}$$

As the singular (when $n \to 1$) part of the $+$ component of the anomalous dimension is !!! $\hat{\gamma}_+ = \hat{\gamma}_{gg} = -8C_A$!!! while the $-$ component does not exist: !!! $(\hat{\gamma}_- = 0)$!!! , we consider below both cases separately.

2.1 The "+" component

The analysis of the "+" component is practically identical to the case studied before. The only difference lies in the appearance of new terms $\varepsilon^+_{ab}(n)$!!! . If they are expanded in the vicinity of $n=1$ in the form $\varepsilon^+_{ab}(n) = \overline{\varepsilon}^+_{ab} + (n-1)\tilde{\varepsilon}^+_{ab}$, !!! then for the terms $\overline{\varepsilon}^+_{ab}$ multiplying $M_b(n,Q^2)$, we have the same results as in previous section:

$$\overline{\varepsilon}^+_{ab} M_b(n,Q^2) \ \stackrel{\mathcal{M}^{-1}}{\longrightarrow}\ \overline{\varepsilon}^+_{ab} A_b I_0(\sigma_{LO}) e^{-\overline{d}_+(1)s_{LO}} \ + \ O(\rho_{LO}),$$

where the symbol $\stackrel{\mathcal{M}^{-1}}{\longrightarrow}$ denotes the inverse Mellin transformation. The values of σ and ρ coincide with those defined in the previous section because $\hat{d}_+ = \hat{d}_{gg}$.

The terms $\tilde{\varepsilon}^{+}_{ab}$ that come with the additional factor $(n-1)$ in front, lead to the following results

$$(n-1)\tilde{\varepsilon}^{+}_{ab}\frac{A_b}{(n-1)}e^{-\hat{d}_{+}s_{LO}/(n-1)} = \tilde{\varepsilon}^{+}_{ab}A_b\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\left(-\hat{d}_{+}s_{LO}\right)^k}{(n-1)^k}$$

$$\xrightarrow{\mathcal{M}^{-1}} \;\; \tilde{\varepsilon}^{+}_{ab}A_b\sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{(k-1)!}\left(-\hat{d}_{+}s_{LO}\right)^k(ln(1/x))^{k-1}$$

$$= \;\; \tilde{\varepsilon}^{+}_{ab}A_b\rho_{LO}I_1(\sigma_{LO}),$$

i.e. the additional factor $(n-1)$ in momentum space leads to replacing the Bessel function $I_0(\sigma_{LO})$ by $\rho_{LO}I_1(\sigma_{LO})$ in x-space.

Thus, we obtain that the term $\varepsilon^{+}_{ab}(n)M_b(n,Q^2)$ leads to the following contribution in x space !!! :

$$(\overline{\varepsilon}^{+}_{ab}I_0(\sigma_{LO}) + \tilde{\varepsilon}^{+}_{ab}\rho_{LO}I_1(\sigma_{LO}))A_b e^{-\overline{d}_{+}(1)s_{LO}} \; + \; O(\rho_{LO})$$

Because the Bessel function $I_\nu(\sigma)$ has the ν-independent asymptotic behavior !!! $e^\sigma/\sqrt{\sigma}$ at $\sigma \to \infty$ (i.e. $x \to 0$), the second term is $O(\rho)$ and must be kept only !!! when $\overline{\varepsilon}^+_{ab} = 0$. This is the case for the quark distribution at the LO approximation.

Using the concrete AD values, one has

$$f_g^+(x,Q^2) \;=\; (A_g + \frac{4}{9}A_q)I_0(\sigma_{LO})e^{-\overline{d}_+(1)s_{LO}} \;+\; O(\rho_{LO}) \quad \text{and}$$
$$f_q^+(x,Q^2) \;=\; \frac{f}{9}(A_g + \frac{4}{9}A_q)\rho_{LO}I_1(\sigma_{LO})e^{-\overline{d}_+(1)s_{LO}} \;+\; O(\rho_{LO})$$

where $\overline{d}_+(1) = 1 + 20f/(27\beta_0)$.

2.2 the "–" component

In this case the anomalous dimension is regular !!! and one has

$$\varepsilon^{-}_{ab}(n) A_b e^{-d_-(n)s} \xrightarrow{\mathcal{M}^{-1}} \overline{\varepsilon}^{-}_{ab}(1) A_b e^{-d_-(1)s_{LO}} + O(x)$$

Using the concrete AD values !!! , we have

$$f^{-}_{g}(x,Q^2) = -\frac{4}{9} A_q e^{-d_-(1)s_{LO}} + O(x) \text{ and}$$

$$f^{-}_{q}(x,Q^2) = A_q e^{-d_-(1)s_{LO}} + O(x),$$

where $d_-(1) = 16f/(27\beta_0)$.

Finally we present the full small x asymptotic results for PD and F_2 structure function at LO of perturbation theory:

$$f_a(x,Q^2) = f_a^+(x,Q^2) + f_a^-(x,Q^2) \text{ and}$$
$$F_2(x,Q^2) = e \cdot f_q(z,Q^2)$$

where f_q^+,f_g^+, f_q^- and f_g^- were already given before and $e = \Sigma_1^f e_i^2/f$ is the average charge square of the f active quarks.

Extension to NLO is trivial and can be found in (A.V.K. and G.Parente, 1998)

A4. Fits of HERA data

At low x, the structure function $F_2(x,Q^2)$ is related to parton densities as (A.V.K. and G.Parente, 1998)

at LO

$$F_2(x,Q^2) = \frac{5}{18} f_q(x,Q^2)$$

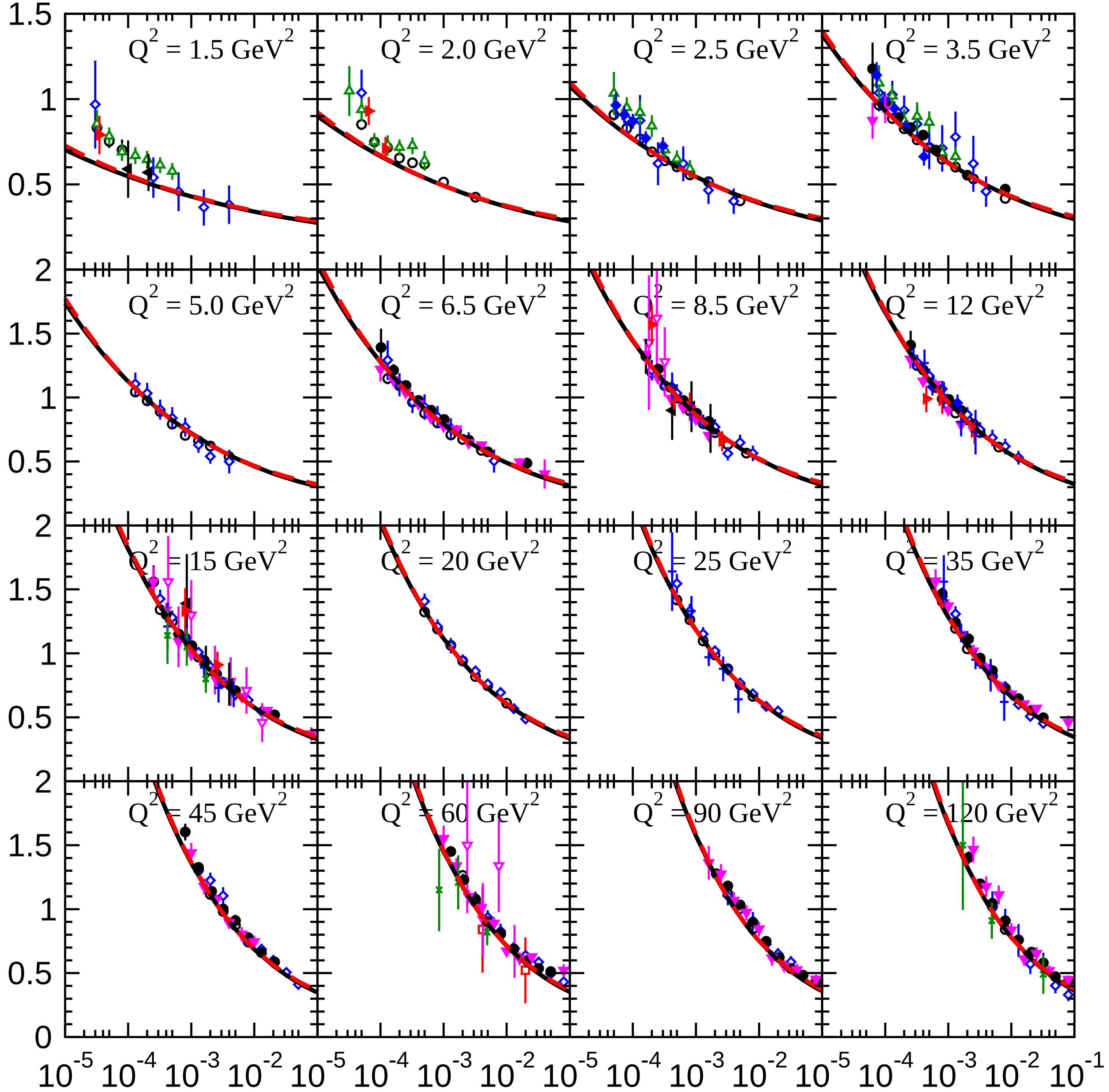
at NLO

$$F_2(x,Q^2) = \frac{5}{18}\left[f_q(x,Q^2) + \frac{2f}{3} a_s(Q^2) f_g(x,Q^2)\right].$$

Fits of HERA experimental data of the structure function $F_2(x,Q^2)$ (A.Yu.Illarionov, A.V.K. and G.Parente, 2004)

!!! Only three parameters: Q_0^2, A_q and A_g

Λ_{QCD} **cannon be extract in small** x **Physics.**



Q^2 = 1.5 GeV^2
Q^2 = 2.0 GeV^2
Q^2 = 2.5 GeV^2
Q^2 = 3.5 GeV^2
Q^2 = 5.0 GeV^2
Q^2 = 6.5 GeV^2
Q^2 = 8.5 GeV^2
Q^2 = 12 GeV^2
Q^2 = 15 GeV^2
Q^2 = 20 GeV^2
Q^2 = 25 GeV^2
Q^2 = 35 GeV^2
Q^2 = 45 GeV^2
Q^2 = 60 GeV^2
Q^2 = 90 GeV^2
Q^2 = 120 GeV^2
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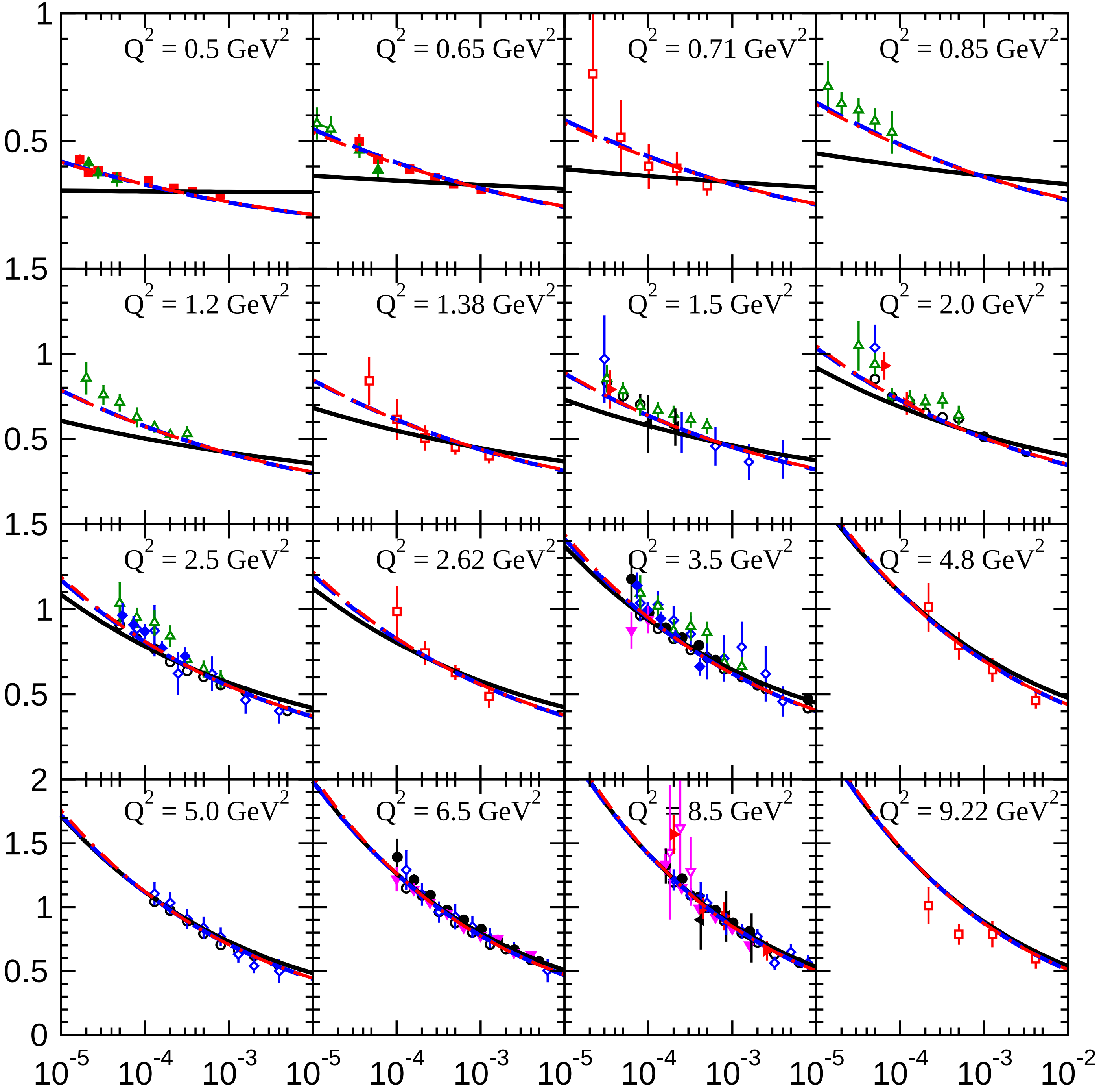
2
10^-5
10^-4
10^-3
10^-2
10^-1



Q^2 = 0.5 GeV^2
Q^2 = 0.65 GeV^2
Q^2 = 0.71 GeV^2
Q^2 = 0.85 GeV^2
Q^2 = 1.2 GeV^2
Q^2 = 1.38 GeV^2
Q^2 = 1.5 GeV^2
Q^2 = 2.0 GeV^2
Q^2 = 2.5 GeV^2
Q^2 = 2.62 GeV^2
Q^2 = 3.5 GeV^2
Q^2 = 4.8 GeV^2
Q^2 = 5.0 GeV^2
Q^2 = 6.5 GeV^2
Q^2 = 8.5 GeV^2
Q^2 = 9.22 GeV^2

A5. Analytical and "frozen" coupling constants

Two modifications of the coupling constant (G.Cvetic, A.Yu.Illarinov, B.A. Kniehl, and A.V.K., 2009); (A.V.K. and B.G. Shaikhatdenov, 2012)

A. More phenomenological.

(G.Curci, M.Greco and Y.Sristava, 1979), (M.Greco, G. Penso and Y.Sristava, 1980), (N.N.Nikolaev and B.M.Zakharov, 1991,1992), (B.Badelek,J.Kwiecinski and A.Stasto, 1997), (A.M.Badalian and Yu.A.Simonov, 1997)

We introduce freezing of the coupling constant by changing its argument $Q^2 \to Q^2 + M_\rho^2$, where M_ρ is sually the ρ-meson mass. Thus, in the formulae of the previous Sections we should do the following replacement

$$a_s(Q^2) \to a_{fr}(Q^2) \equiv a_s(Q^2 + M_\rho^2) \qquad (6)$$

B. Theoretical approach.

Incorporates the Shirkov-Solovtsov idea (D.V.Shirkov and L.I.Solovtsov, 1997), about analyticity of the coupling constant that leads to the additional its power dependence.

(K.A.Milton, A.V. Nesterenko, O.Solovtsova, G. Cvetic, C. Valenzuela, I. Schmidt, O. Teryaev, N. Stefanis, A. Bakulev, S. Mikhailov, ...)

Then, in the formulae of the previous Section the coupling constant $a_s(Q^2)$ should be replaced as follows

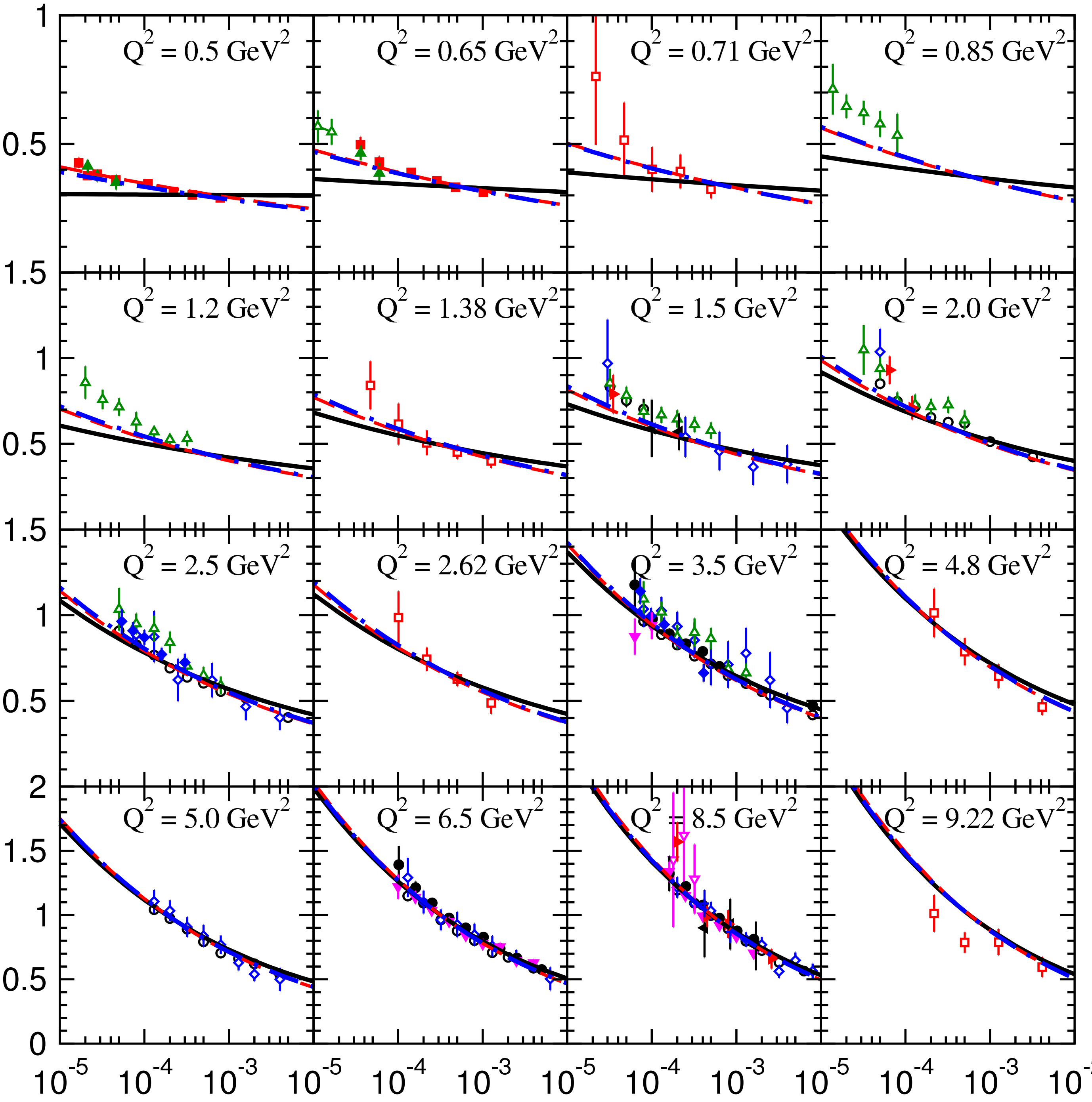
$$a_{an}^{LO}(Q^2) = a_s(Q^2) - \frac{1}{\beta_0}\frac{\Lambda_{LO}^2}{Q^2 - \Lambda_{LO}^2} \tag{7}$$

at the LO approximation and

$$a_{an}(Q^2) = a_s(Q^2) - \frac{1}{2\beta_0}\frac{\Lambda^2}{Q^2 - \Lambda^2} + ... \tag{8}$$

at the NLO approximation, where the symbol ... marks numerically small terms.

The replacement (7) and (8) is applicable only for ruther large values of Q^2!!!
For lower Q^2 values it is better to use the fraction analytic perturbation theory (A. Bakulev, S. Mikhailov, N. Stefanis, 2005) , but it direct application is ruther difficult.



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Q^2 = 8.5 GeV^2
Q^2 = 9.22 GeV^2
0
0.5
1
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10^-4
10^-3
10^-2

Table 1: The result of the LO and NLO fits to H1 and ZEUS data for different low Q^2 cuts. In the fits f is fixed to 4 flavors.

	A_g	A_q	Q_0^2 [GeV^2]	$\chi^2/n.o.p.$
$Q^2 \geq 1.5\mathrm{GeV}^2$				
LO	0.784±.016	0.801±.019	0.304±.003	754/609
LO&an.	0.932±.017	0.707±.020	0.339±.003	632/609
LO&fr.	1.022±.018	0.650±.020	0.356±.003	547/609
NLO	-0.200±.011	0.903±.021	0.495±.006	798/609
NLO&an.	0.310±.013	0.640±.022	0.702±.008	655/609
NLO&fr.	0.180±.012	0.780±.022	0.661±.007	669/609
$Q^2 \geq 0.5\mathrm{GeV}^2$				
LO	0.641±.010	0.937±.012	0.295±.003	1090/662
LO&an.	0.846±.010	0.771±.013	0.328±.003	803/662
LO&fr.	1.127±.011	0.534±.015	0.358±.003	679/662
NLO	-0.192±.006	1.087±.012	0.478±.006	1229/662
NLO&an.	0.281±.008	0.634±.016	0.680±.007	633/662
NLO&fr.	0.205±.007	0.650±.016	0.589±.006	670/662

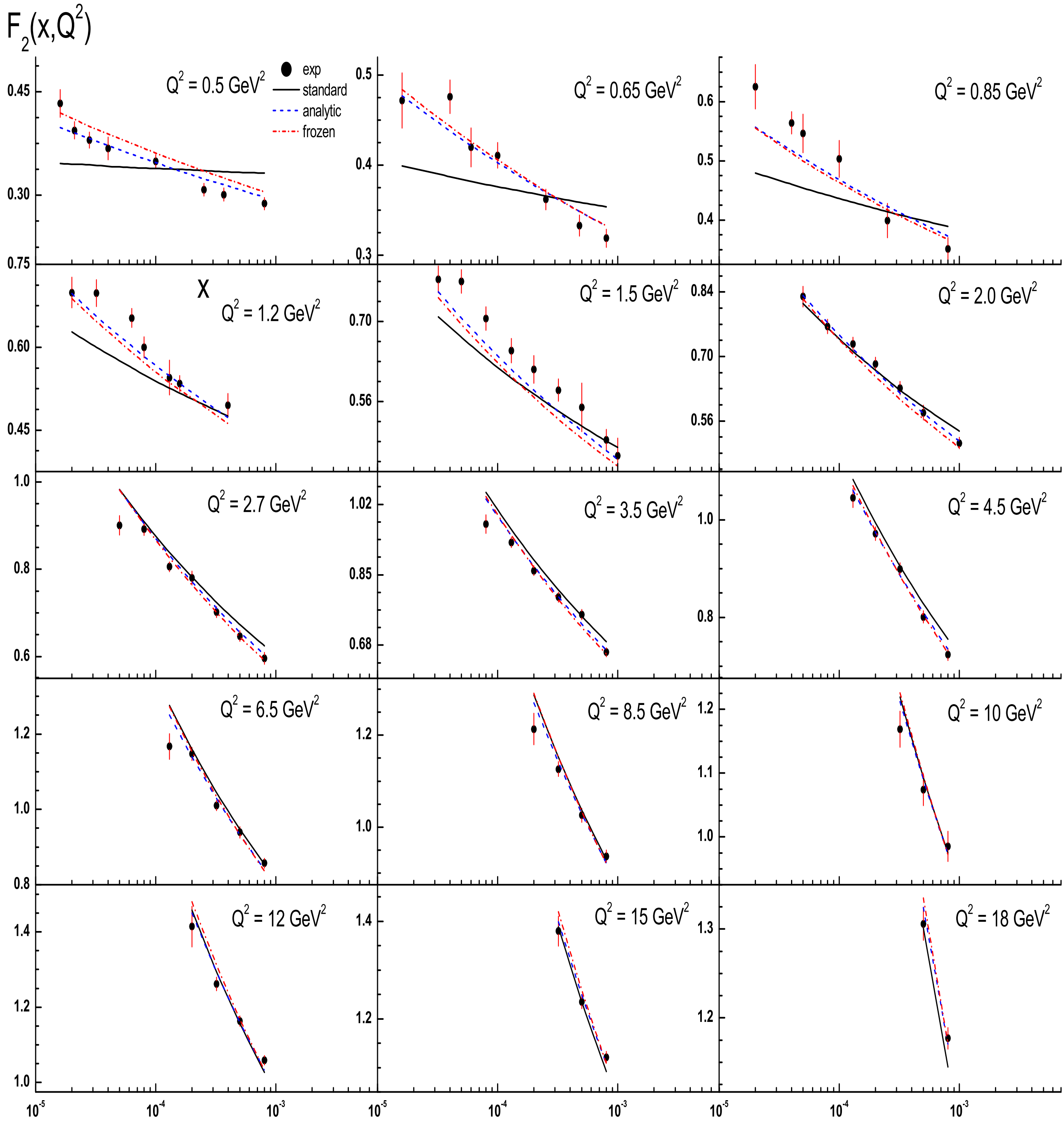
- Usage of the analytical and "frozen" coupling constants leads to improvement with data: χ^2 decreased twicely
- Really, no difference between results based on the analytical and "frozen" coupling constants.

!!! One example of application the analytical and "frozen" coupling constants: (A.V.Kotikov, A.V.Lipatov and N.P.Zotov, 2004)

New $H1\&ZEUS$ (2010) experimental data for F_2:
(F.D. Aaron *et al.*, 2010)
there is a good agreement for $Q^2 \geq 0.5$ GeV^2.

Table 2: The results of LO and NLO fits to H1 & ZEUS data with various lower cuts on Q^2; in the fits the r
f is fixed to 4.

	A_g	A_q	Q_0^2 [GeV^2]	$\chi^2/n.d.f.$
$Q^2 \geq 5\mathrm{GeV}^2$				
LO	0.623±0.055	1.204±0.093	0.437±0.022	1.00
LO&an.	0.796±0.059	1.103±0.095	0.494±0.024	0.85
LO&fr.	0.782±0.058	1.110±0.094	0.485±0.024	0.82
NLO	-0.252±0.041	1.335±0.100	0.700±0.044	1.05
NLO&an.	0.102±0.046	1.029±0.106	1.017±0.060	0.74
NLO&fr.	-0.132±0.043	1.219±0.102	0.793±0.049	0.86
$Q^2 \geq 3.5\mathrm{GeV}^2$				
LO	0.542±0.028	1.089±0.055	0.369±0.011	1.73
LO&an.	0.758±0.031	0.962±0.056	0.433±0.013	1.32
LO&fr.	0.775±0.031	0.950±0.056	0.432±0.013	1.23
NLO	-0.310±0.021	1.246±0.058	0.556±0.023	1.82
NLO&an.	0.116±0.024	0.867±0.064	0.909±0.330	1.04
NLO&fr.	-0.135±0.022	1.067±0.061	0.678±0.026	1.27
$Q^2 \geq 2.5\mathrm{GeV}^2$				
LO	0.526±0.023	1.049±0.045	0.352±0.009	1.87
LO&an.	0.761±0.025	0.919±0.046	0.422±0.010	1.38
LO&fr.	0.794±0.025	0.900±0.047	0.425±0.010	1.30
NLO	-0.322±0.017	1.212±0.048	0.517±0.018	2.00
NLO&an.	0.132±0.020	0.825±0.053	0.898±0.026	1.09
NLO&fr.	-0.123±0.018	1.016±0.051	0.658±0.021	1.31
$Q^2 \geq 0.5\mathrm{GeV}^2$				
LO	0.366±0.011	1.052±0.016	0.295±0.005	5.74
LO&an.	0.665±0.012	0.804±0.019	0.356±0.006	3.13
LO&fr.	0.874±0.012	0.575±0.021	0.368±0.006	2.96
NLO	-0.443±0.008	1.260±0.012	0.387±0.010	6.62
NLO&an.	0.121±0.008	0.656±0.024	0.764±0.015	1.84
NLO&fr.	-0.071±0.007	0.712±0.023	0.529±0.011	2.79

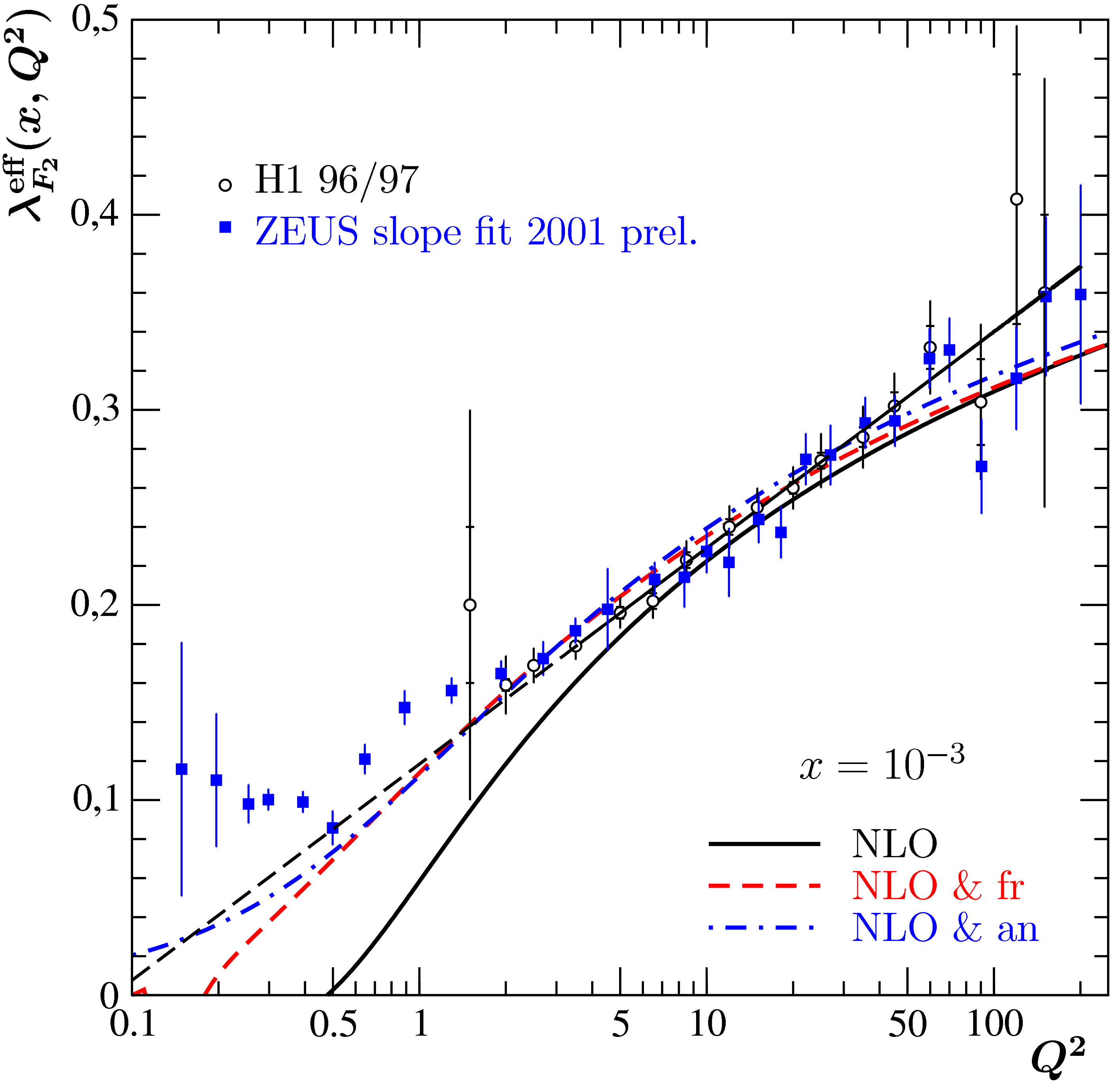


$F_2(x,Q^2)$
exp
standard
analytic
frozen
Q^2 = 0.5 GeV^2
Q^2 = 0.65 GeV^2
Q^2 = 0.85 GeV^2
Q^2 = 1.2 GeV^2
Q^2 = 1.5 GeV^2
Q^2 = 2.0 GeV^2
Q^2 = 2.7 GeV^2
Q^2 = 3.5 GeV^2
Q^2 = 4.5 GeV^2
Q^2 = 6.5 GeV^2
Q^2 = 8.5 GeV^2
Q^2 = 10 GeV^2
Q^2 = 12 GeV^2
Q^2 = 15 GeV^2
Q^2 = 18 GeV^2
X
10^{-5}
10^{-4}
10^{-3}

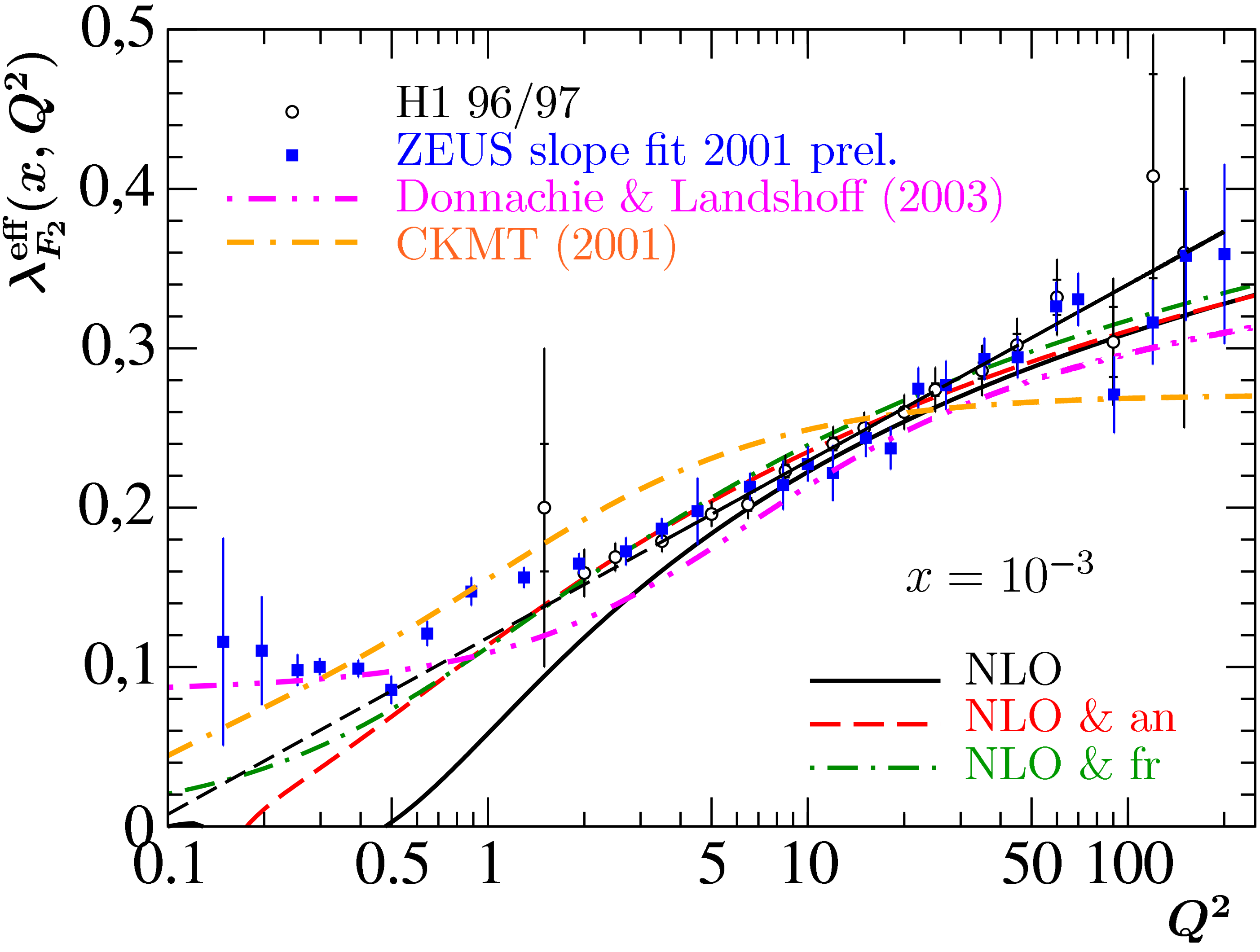
The results for F_2 and for the slope of the SF F_2

The double-logarithmic behaviour can mimic a power law shape over a limited region of x, Q^2.

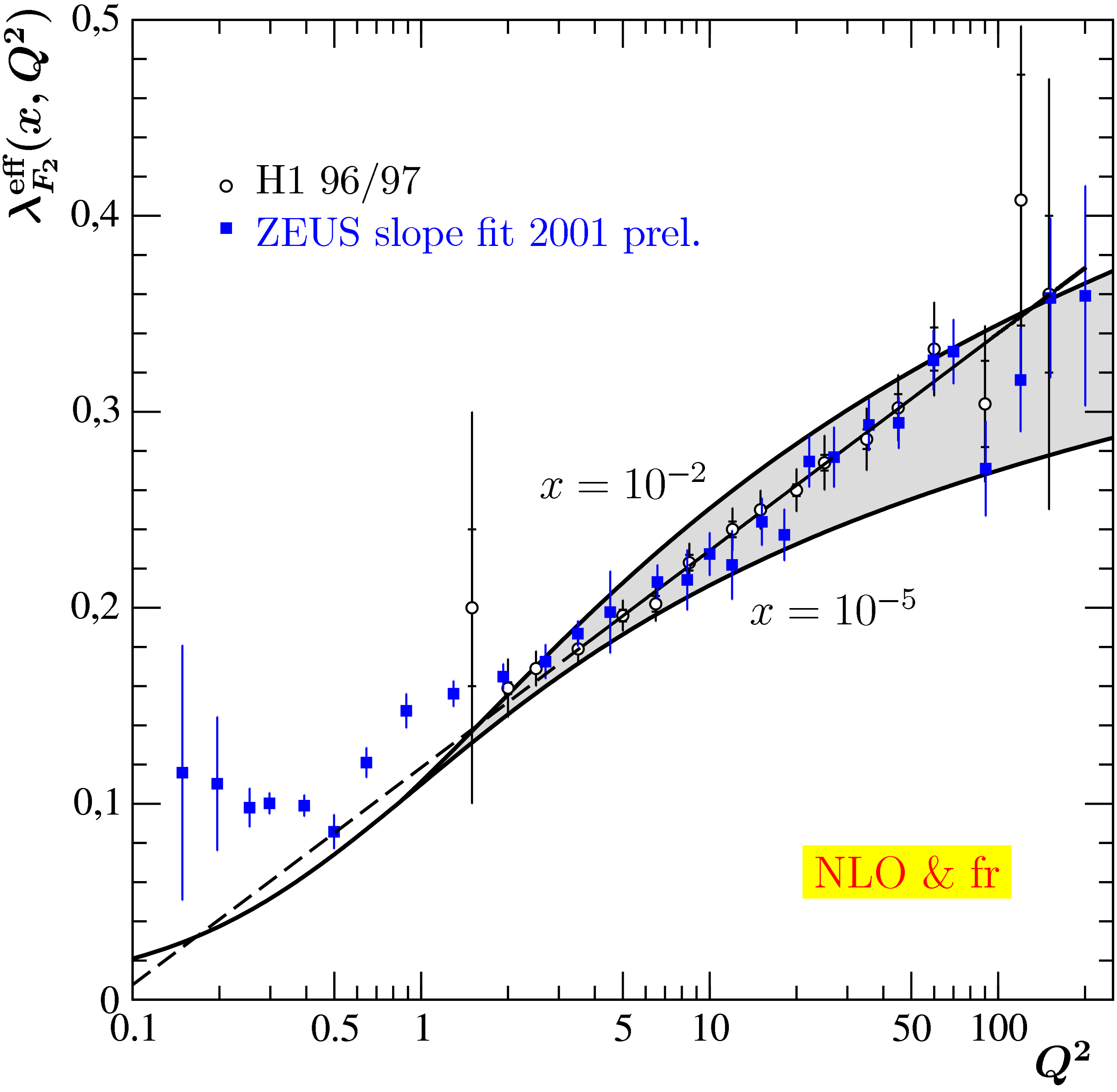
$$f_a(x,Q^2) \sim x^{-\lambda_a^{eff}(x,Q^2)} \text{ and } F_2(x,Q^2) \sim x^{-\lambda_{F2}^{eff}(x,Q^2)}$$



$\lambda^{\text{eff}}_{F_2}(x, Q^2)$
0,5
0,4
0,3
0,2
0,1
0
H1 96/97
ZEUS slope fit 2001 prel.
$x = 10^{-3}$
NLO
NLO & fr
NLO & an
0.1
0.5
1
5
10
50
100
Q^2



$\lambda^{\mathrm{eff}}_{F_2}(x, Q^2)$
H1 96/97
ZEUS slope fit 2001 prel.
Donnachie & Landshoff (2003)
CKMT (2001)
$x = 10^{-3}$
NLO
NLO & an
NLO & fr
0,5
0,4
0,3
0,2
0,1
0
0.1
0.5
1
5
10
50
100
Q^2



$\lambda^{\mathrm{eff}}_{F_2}(x, Q^2)$
0,5
0,4
0,3
0,2
0,1
0
H1 96/97
ZEUS slope fit 2001 prel.
$x = 10^{-2}$
$x = 10^{-5}$
NLO & fr
0.1
0.5
1
5
10
50
100
Q^2

A6. Conclusion

- I have demonstrated the low x asymptotics of parton densities and SF F_2.
- Low x asymptotics of F_2 are in good agreement with data from HERA at $Q^2 \geq 2.5$ GeV^2.
- Usage of the analytical and "frozen" coupling constants leads to improvement with data from HERA at $Q^2 \leq 2.5$ GeV^2, including the new H1+ZEUS data for F_2.
  (F.D. Aaron *et al.*, 2010).

Next steps:

- To add the NNLO corrections (which has $\sim 1/(n-1)^2$ poles at $n \to 1)$. So, the NNLO small-x asyptotics $\sim exp[\sim (\ln(1/x))^{2/3}]$ is more singular then the corresponding LO and NLO ones $\sim exp[\sim \sqrt{\ln(1/x)}]$.
- To consider the new H1+ZEUS data for F_2^c. (H. Abramowitz *et al.*, 2012).

B1. Average Multiplicities (absract)

- I present the new results (B.Bolzoni, B.A. Kniehl and A.V.K., 2013) for gluon and quark average multiplicities, which are motivated by recent progress in timelike small-x resummation obtained in the $\overline{\mathrm{MS}}$ scheme. (C.-H.Korn, A. Vogt and K.Yeats, 2012).

  The results contain the next-to-next-to-leading-logarithmic (NNLL) resummed expressions and depend on two nonperturbative parameters with clear and simple physical interpretations.

- We did a global fit of these two quantities. Our results solved a longstandig problem of QCD: a disagreement between theoretical predictions for the ration of gluon and quark average multiplicities and the corresponding experimental data.

- We finally proposed also to use the multiplicity data as a new way to extract the strong-coupling constant. We obtained $\alpha_s^{(5)}(M_z) = 0.1199 \pm 0.0026$ in the $\overline{\mathrm{MS}}$ scheme in an approximation equivalent to next-to-next-to-leading order (NNNLO) enhanced by the resummations of $\ln x$ terms through the NNLL level, in excellent agreement with the present world average.

B2. Introduction II

When jets are produced at colliders, they can be initiated either by a quark or a gluon. The two types of jets are expected to exhibit different properties.

The production of hadrons is a typical process where nonperturbative phenomena are involved.

However, for particular observables, this problem can be avoided. In particular, the *counting* of hadrons. In this case, one can adopt with quite high accuracy *the hypothesis of Local Parton-Hadron Duality (LPHD)*: **parton distributions are renormalized in the hadronization process without changing their shapes** (Ya.I.Azimov, Yu.L.Dokshitzer, V.A.Khoze and S.I.Troyan, 1985). Hence, if the scale Q is large enough, perturbative QCD privides predictions without an usage of phenomenological models of hadronization.

However, **the computation of average jet multiplicities indeed requires small-x resummation**, (A.H.Mueller, 1981) It was shown that the singularities for $x \sim 0$, which are encoded in large logarithms of the kind $\ln^k(1/x)$ and disappear after resummation. Usually, resummation includes the singularities from all orders according to a certain logarithmic accuracy, for which it $restores$ perturbation theory.

Example, in the Mellin space, N is Mellin moment, $\omega = N - 1$:

$$a_s \left(\frac{1}{\omega} + Const\right) \to a_s \left(\frac{1}{\omega_{\rm eff}} + Const\right)$$

$$\frac{a_s}{\omega} \to \frac{a_s}{\omega_{\rm eff}} = \frac{a_s}{\omega} \frac{1}{\sqrt{1 + \frac{8C_A a_s}{\omega^2}}} \to \sqrt{\frac{a_s}{8C_A}} \quad (\text{at} \;\; \omega \to 0)\,,$$

i.e.

$$\omega \to \omega_{\rm eff} = \omega \sqrt{1 + \frac{8C_A a_s}{\omega^2}} \to \sqrt{8C_A a_s} \quad (\text{at} \;\; \omega \to 0)\,.$$

!!!So, after resummation we have perturbation theory in the parameter $\sqrt{a_s}$, not a_s.!!!

Small-x resummation has recently been carried out for time-like splitting fuctions in the $\overline{\mathrm{MS}}$ scheme at the next-to-leading-logarithmic (NLL) level of accuracy (A.Vogt, 2011). and at the next-to-next-to-leading-logarithmic (NNLL) level. (C.-H.Korn, A. Vogt and K.Yeats, 2012).

Thanks to these results, we are able to analytically compute the NNLL contributions to the evolutions of the average gluon and quark jet multiplicities.

B3. Fragmentation functions and their evolution

The evolution of the fragmentation functions $D_a(x, \mu^2)$ for the gluon–quark-singlet system $a = g, s$. In Mellin space, is:

$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} D_s(\omega, \mu^2) \\ D_g(\omega, \mu^2) \end{pmatrix} = \begin{pmatrix} P_{qq}(\omega, a_s) & P_{gq}(\omega, a_s) \\ P_{qg}(\omega, a_s) & P_{gg}(\omega, a_s) \end{pmatrix} \begin{pmatrix} D_s(\omega, \mu^2) \\ D_g(\omega, \mu^2) \end{pmatrix}, \tag{9}$$

where $P_{ij}(\omega, a_s)$, with $i, j = g, q$, are the timelike splitting functions, $\omega = N - 1$, with N being the standard Mellin moments with respect to x, and $a_s(\mu^2) = \alpha_s(\mu)/(4\pi)$ is the couplant.

The standard definition of the hadron average multiplicities in terms of the fragmentation functions is given by their integral over x, which corresponds to the first Mellin moment, with $\omega = 0$:

$$\langle n_h(Q^2)\rangle_a \equiv \left[\int_0^1 dx\, x^\omega D_a(x,Q^2)\right]_{\omega=0} = D_a(\omega=0,Q^2) \quad (10)$$

The timelike splitting functions $P_{ij}(\omega, a_s)$ may be computed perturbatively in a_s,

$$P_{ij}(\omega, a_s) = \sum_{k=0}^{\infty} a_s^{k+1} P_{ij}^{(k)}(\omega). \quad (11)$$

The functions $P_{ij}^{(k)}(\omega)$ for $k = 0, 1, 2$ in the $\overline{\mathrm{MS}}$ scheme may be found through NNLO and with small-x resummation through NNLL accuracy.

B4. Diagonalization

It is not in general possible to diagonalize Eq. (9) because the contributions to the timelike-splitting-function matrix do not commute at different orders.

The *usual approach* is then to write a series expansion about the leading-order (LO) solution, which can in turn be diagonalized. One thus starts by choosing a basis in which the timelike-splitting-function matrix is diagonal at LO

$$
\begin{aligned}
P(\omega, a_s) &= \begin{pmatrix} P_{++}(\omega, a_s) & P_{-+}(\omega, a_s) \\ P_{+-}(\omega, a_s) & P_{--}(\omega, a_s) \end{pmatrix} \\
&= a_s \begin{pmatrix} P_{++}^{(0)}(\omega) & 0 \\ 0 & P_{--}^{(0)}(\omega) \end{pmatrix} + a_s^2 P^{(1)}(\omega) + O(a_s^3), \qquad (12)
\end{aligned}
$$

with eigenvalues $P_{\pm\pm}^{(0)}(\omega)$.

It is convenient to represent the change of basis for the fragmentation functions order by order for $k \geq 0$:

$$\begin{aligned} D^+(\omega, \mu_0^2) &= (1-\alpha_\omega) D_s(\omega, \mu_0^2) - \epsilon_\omega D_g(\omega, \mu_0^2), \\ D^-(\omega, \mu_0^2) &= \alpha_\omega D_s(\omega, \mu_0^2) + \epsilon_\omega D_g(\omega, \mu_0^2). \end{aligned} \tag{13}$$

This implies for the components of the timelike-splitting-function matrix that

$$\begin{aligned} P_{--}^{(k)}(\omega) &= \alpha_\omega P_{qq}^{(k)}(\omega) + \epsilon_\omega P_{qg}^{(k)}(\omega) + \beta_\omega P_{gq}^{(k)}(\omega) + (1-\alpha_\omega) P_{gg}^{(k)}(\omega), \\ P_{-+}^{(k)}(\omega) &= P_{--}^{(k)}(\omega) - \left(P_{qq}^{(k)}(\omega) + \frac{1-\alpha_\omega}{\epsilon_\omega} P_{gq}^{(k)}(\omega) \right), \\ P_{++}^{(k)}(\omega) &= P_{qq}^{(k)}(\omega) + P_{gg}^{(k)}(\omega) - P_{--}^{(k)}(\omega), \\ P_{+-}^{(k)}(\omega) &= P_{++}^{(k)}(\omega) - \left(P_{qq}^{(k)}(\omega) - \frac{\alpha_\omega}{\epsilon_\omega} P_{gq}^{(k)}(\omega) \right) \\ &= P_{gg}^{(k)}(\omega) - \left(P_{--}^{(k)}(\omega) - \frac{\alpha_\omega}{\epsilon_\omega} P_{gq}^{(k)}(\omega) \right), \end{aligned} \tag{14}$$

where the elements of the matrix for diagonalization **(LO projectors !!!)**

$$\alpha_\omega = \frac{P_{qq}^{(0)}(\omega) - P_{++}^{(0)}(\omega)}{P_{--}^{(0)}(\omega) - P_{++}^{(0)}(\omega)}, \quad \epsilon_\omega = \frac{P_{gq}^{(0)}(\omega)}{P_{--}^{(0)}(\omega) - P_{++}^{(0)}(\omega)},$$

$$\beta_\omega = \frac{P_{qg}^{(0)}(\omega)}{P_{--}^{(0)}(\omega) - P_{++}^{(0)}(\omega)}. \tag{15}$$

We make the following ansatz to expand about the diagonal part of the timelike-splitting-function matrix in the plus-minus basis: (similary to (A.Buras, 1980) in spacelike case)

$$T_{\mu^2}\left\{\exp\int_{\mu_0^2}^{\mu^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}P(\bar{\mu}^2)\right\} = Z^{-1}(\mu^2)\exp\left[\int_{\mu_0^2}^{\mu^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}P^D(\bar{\mu}^2)\right]Z(\mu_0^2), \tag{16}$$

where

$$P^D(\omega) = \begin{pmatrix} P_{++}(\omega) & 0 \\ 0 & P_{--}(\omega) \end{pmatrix} \tag{17}$$

is the diagonal part and Z is a matrix in the plus-minus basis which has a perturbative expansion of the form

$$Z(\mu^2) = 1 + a_s(\mu^2)Z^{(1)} + O(a_s^2). \tag{18}$$

After some algebra we find:

$$Z^{(1)}_{\pm\pm}(\omega) = 0, \qquad Z^{(1)}_{\pm\mp}(\omega) = \frac{P^{(1)}_{\pm\mp}(\omega)}{\beta_0 + P^{(0)}_{\pm\pm}(\omega) - P^{(0)}_{\mp\mp}(\omega)}. \tag{19}$$

So, we can write the gluon and quark-singlet fragmentation functions in the following way:

$$D_a(\omega,\mu^2) \equiv D_a^+(\omega,\mu^2) + D_a^-(\omega,\mu^2), \qquad a = g, s, \tag{20}$$

where $D_a^+(\omega,\mu^2)$ evolves like a plus component and $D_a^-(\omega,\mu^2)$ like a minus component.

Finally, we find that

$$D_a^{\pm}(\omega,\mu^2) = \tilde{D}_a^{\pm}(\omega,\mu_0^2)\hat{T}_{\pm}(\omega,\mu^2,\mu_0^2)\, H_a^{\pm}(\omega,\mu^2), \tag{21}$$

where

$$\hat{T}_{\pm}(\omega,\mu^2,\mu_0^2) = \exp\left[\int_{a_s(\mu_0^2)}^{a_s(\mu^2)} \frac{d\bar{a}_s}{\beta(\bar{a}_s)}\, P_{\pm\pm}(\omega,\bar{a}_s)\right]. \tag{22}$$

has a RG-type exponential form and

$$\tilde{D}_g^+(\omega,\mu_0^2) = -\frac{\alpha_\omega}{\epsilon_\omega}\tilde{D}_s^+(\omega,\mu_0^2), \qquad \tilde{D}_g^-(\omega,\mu_0^2) = \frac{1-\alpha_\omega}{\epsilon_\omega}\tilde{D}_s^-(\omega,\mu_0^2),$$
$$\tilde{D}_s^+(\omega,\mu_0^2) = \tilde{D}^+(\omega,\mu_0^2), \qquad \tilde{D}_s^-(\omega,\mu_0^2) = \tilde{D}^-(\omega,\mu_0^2), \tag{23}$$

(the ratios $\tilde{D}_g^\pm(\omega,\mu_0^2)/\tilde{D}_s^\pm(\omega,\mu_0^2$ have only LO results !!!)

with $H_a^\pm(\omega,\mu^2)$ are perturbative functions given by

$$H_a^\pm(\omega,\mu^2) = 1 - a_s(\mu^2)Z^{(1)}_{\pm\mp,a}(\omega) + O(a_s^2). \tag{24}$$

and

$$Z^{(1)}_{\pm\mp,g}(\omega) = -Z^{(1)}_{\pm\mp}(\omega)\left(\frac{1-\alpha_\omega}{\alpha_\omega}\right)^{\pm 1}, \qquad Z^{(1)}_{\pm\mp,s}(\omega) = Z^{(1)}_{\pm\mp}(\omega), \tag{25}$$

where $Z^{(1)}_{\pm\mp}(\omega)$ is given by Eq. (19).

B5. Resummation

Reliable computations of average jet multiplicities require resummed analytic expressions for the splitting functions because one has to evaluate the first Mellin moment (corresponding to $\omega = N - 1 = 0$), which is a divergent quantity in the fixed-order perturbative approach. As is well known, resummation overcomes this problem, as demonstrated in the pioneering works by Mueller (A.H.Mueller, 1981) and others (B.I.Ermolaev and V.S.Fadin, 1981), (Yu.L.Dokshitzer, V.S Fadin and V.A.Khoze, 1982,1983)

For future considerations, we remind the reader of an assumpion already made (S.Albino, B.Bolzoni, B.A. Kniehl and A.V.K., 2012) according to which the splitting functions $P_{--}^{(k)}(\omega)$ and $P_{+-}^{(k)}(\omega)$ are supposed to be free of singularities in the limit $\omega \to 0$. In fact, this is expected to be true to all orders. This is certainly true at the LL and NLL levels for the timelike splitting functions, as was verified in (S.Albino, B.Bolzoni, B.A. Kniehl and A.V.K., 2012). This is also true at the NNLL level, as may be explicitly checked by inserting the results of (C.-H.Korn, A. Vogt and K.Yeats, 2012).

So, the minus components are devoid of singularities as $\omega \to 0$ and thus are not resummed.

In order to be able to obtain the average jet multiplicities, we have to first evaluate the first Mellin momoments of the timelike splitting functions in the plus-minus basis. According to Eq. (14) together with the results given in (A.H. Mueller, 1981), (C.-H.Korn, A. Vogt and K.Yeats, 2012) we have

$$P_{++}^{\mathrm{NNLL}}(\omega=0)=\gamma_0(1-K_1\gamma_0+K_2\gamma_0^2), \tag{26}$$

where

$$\begin{aligned}
\gamma_0 &= P_{++}^{\mathrm{LL}}(\omega=0)=\sqrt{2C_A a_s}, &(27)\\
K_1 &= \frac{1}{12}\left[11+4\frac{n_f T_R}{C_A}\left(1-\frac{2C_F}{C_A}\right)\right], &(28)\\
K_2 &= \frac{1}{288}\left[1193-576\zeta_2-56\frac{n_f T_R}{C_A}\left(5+2\frac{C_F}{C_A}\right)\right]\\
&\quad +16\frac{n_f^2 T_R^2}{C_A^2}\left(1+4\frac{C_F}{C_A}-12\frac{C_F^2}{C_A^2}\right), &(29)
\end{aligned}$$

and

$$P_{-+}^{\mathrm{NNLL}}(\omega = 0) = -\frac{C_F}{C_A}\, P_{qg}^{NNLL}(\omega = 0), \tag{30}$$

where

$$P_{qg}^{\mathrm{NNLL}}(\omega = 0) = \frac{16}{3} n_f T_R a_s \\ -\frac{2}{3} n_f T_R \left[17 - 4\,\frac{n_f T_R}{C_A}\left(1 - \frac{2C_F}{C_A}\right)\right]\left(2C_A a_s^3\right)^{1/2}. \tag{31}$$

For the P_{+-} component, we obtain

$$P_{+-}^{\mathrm{NNLL}}(\omega = 0) = O(a_s^2). \tag{32}$$

Finally, as for the P_{--} component, we note that its LO expression produces a finite, nonvanishing term for $\omega = 0$ that is of the same order in a_s as the NLL-resummed results in Eq. (26), which leads us to use the following expression for the P_{--} component:

$$P_{--}^{\mathrm{NNLL}}(\omega = 0) = -\frac{8n_f T_R C_F}{3C_A}\, a_s + O(a_s^2), \tag{33}$$

at NNLL accuracy.

We can now perform the integration in Eq. (22) through the NNLL level, which yields

$$\begin{aligned}
\hat{T}_{\pm}^{\mathrm{NNLL}}(0,Q^2,Q_0^2) &= \frac{T_{\pm}^{\mathrm{NNLL}}(Q^2)}{T_{\pm}^{\mathrm{NNLL}}(Q_0^2)}, \\
T_{+}^{\mathrm{NNLL}}(Q^2) &= \exp\left\{\frac{4C_A}{\beta_0\gamma^0(Q^2)}\left[1+(b_1-2C_AK_2)\,a_s(Q^2)\right]\right\}\Big(a_s(Q^2)\Big)^{d_+}, \\
T_{-}^{\mathrm{NNLL}}(Q^2) &= T_{-}^{\mathrm{NLL}}(Q^2) = \Big(a_s(Q^2)\Big)^{d_-}, \qquad (34)
\end{aligned}$$

where

$$b_1 = \frac{\beta_1}{\beta_0}, \qquad d_- = \frac{8n_fT_RC_F}{3C_A\beta_0}, \qquad d_+ = \frac{2C_AK_1}{\beta_0}. \qquad (35)$$

B6. Multiplicities

We are now ready to define the average gluon and quark jet multiplicities in our formalism, namely

$$\langle n_h(Q^2)\rangle_a \equiv D_a(0,Q^2) = D_a^+(0,Q^2) + D_a^-(0,Q^2), \qquad (36)$$

with $a = g, s$, respectively.

On the other hand, from Eqs. (21) and (23), it follows that

$$r_+(Q^2) \equiv \frac{D_g^+(0,Q^2)}{D_s^+(0,Q^2)} = -\lim_{\omega\to 0} \frac{\alpha_\omega}{\epsilon_\omega} \frac{H_g^+(\omega,Q^2)}{H_s^+(\omega,Q^2)}, \qquad (37)$$

$$r_-(Q^2) \equiv \frac{D_g^-(0,Q^2)}{D_s^-(0,Q^2)} = \lim_{\omega\to 0} \frac{1-\alpha_\omega}{\epsilon_\omega} \frac{H_g^-(\omega,Q^2)}{H_s^-(\omega,Q^2)}. \qquad (38)$$

Using these definitions and again Eq. (21), we may write general expressions for the average gluon and quark jet multiplicities:

$$\begin{aligned}
\langle n_h(Q^2)\rangle_g &= \tilde{D}_g^+(0,Q_0^2)\hat{T}_+^{\mathrm{res}}(0,Q^2,Q_0^2)H_g^+(0,Q^2) \\
&\quad + \tilde{D}_s^-(0,Q_0^2)r_-(Q^2)\hat{T}_-^{\mathrm{res}}(0,Q^2,Q_0^2)H_s^-(0,Q^2), \\
\langle n_h(Q^2)\rangle_s &= \frac{\tilde{D}_g^+(0,Q_0^2)}{r_+(Q^2)}\hat{T}_+^{\mathrm{res}}(0,Q^2,Q_0^2)H_g^+(0,Q^2) \\
&\quad + \tilde{D}_s^-(0,Q_0^2)\hat{T}_-^{\mathrm{res}}(0,Q^2,Q_0^2)H_s^-(0,Q^2). \qquad (39)
\end{aligned}$$

At the LO in a_s, the coefficients of the RG exponents are given by

$$\begin{aligned}
r_+(Q^2) &= \frac{C_A}{C_F}, \qquad r_-(Q^2) = 0, \\
H_s^\pm(0,Q^2) &= 1, \qquad \tilde{D}_a^\pm(0,Q_0^2) = D_a^\pm(0,Q_0^2), \qquad (40)
\end{aligned}$$

for $a = g, s$.

It would, of course, be desirable to include higher-order corrections in Eqs. (40). However, this is highly nontrivial because the general perturbative structures of the functions $H_a^{\pm}(\omega,\mu^2)$ and $Z_{\pm\mp,a}(\omega,a_s)$, which would allow us to resum those higher-order corrections, are presently unknown.

On the one hand, **it is well-known that the plus components by themselves represent the dominant contributions to both the average gluon and quark jet multiplicities** (see, e.g., (M.Schmelling, 1995) for the gluon case and (I.M.Dremin and J.W.Gary, 2001) for the quark case).

On the other hand, Eq. (38) tells us that $D_g^-(0, Q^2)$ **is suppressed with respect to** $D_s^-(0, Q^2)$ **because** $\alpha_\omega \sim 1 + \mathcal{O}(\omega)$**. So, keeping** $r_-(Q^2) = 0$ **also beyond LO should represent a good approximation.**

Nevertheless, we shall explain below how to obtain the first nonvanishing contribution to $r_-(Q^2)$.

Furthermore, we notice that **higher-order corrections to** $H_a^\pm(0, Q^2)$ and $\tilde{D}_a^\pm(0, Q_0^2)$ just represent redefinitions of $D_a^\pm(0, Q_0^2)$ by constant factors apart from running-coupling effects. Therefore, we assume that these corrections **can be neglected**.

We now discuss higher-order corrections to $r_+(Q^2)$. As already mentioned above, we introduced (B.Bolzoni, B.A. Kniehl and A.V.K., 2013) an effective approach to perform the resummation of the first Mellin moment of the "plus" component of the anomalous dimension. In that approach, **resummation is performed by taking the fixed-order plus component and substituting $\omega = \omega_{\rm eff}$** , where

$$\omega_{\rm eff} = 2\sqrt{2C_A a_s} = 2\gamma_0. \tag{41}$$

We now show that this approach is exact to $O(\sqrt{a_s})$. We indeed recover Eq. (27) by substituting $\omega = \omega_{\rm eff}$ in the leading singular term of the LO splitting function $P_{++}(\omega, a_s)$,

$$P_{++}^{\rm LO}(\omega) = \frac{4C_A a_s}{\omega} + O(\omega^0). \tag{42}$$

We may then also substitute $\omega = \omega_{\rm eff}$ in the LO result for $r_+(Q^2)$ (i.e. to Eq. (37)). We thus find

$$r_+(Q^2) = \frac{C_A}{C_F}\left[1 - \frac{\sqrt{2a_s(Q^2)C_A}}{3}\left(1 + 2\frac{n_f T_R}{C_A} - 4\frac{C_F n_f T_R}{C_A^2}\right)\right] + O(a_s), \tag{43}$$

which coincides with the result obtained by Mueller (A.H.Mueller, 1984).

For this reason and because (I.M.Dremin and J.W.Gary, 1999) the average gluon and quark jet multiplicities evolve with only one RG exponent **(only with "plus" component)**, we can intepete the result of (A.Capella, I.M.Dremin, J.W.Gary, V.A. Nechitailo and J. Tran Thanh Van, 2000) as higher-order corrections to Eq. (43).

So, we use the results of (A.Capella, I.M.Dremin, J.W.Gary, V.A. Nechitailo and J. Tran Thanh Van, 2000) for the ratio gluon and quark average multiplicities as the estimation for $r_+(Q^2)$.

Since we showed that our approach reproduces exact analytic results at $O(\sqrt{a_s})$, we may safely apply it to predict the first non-vanishing correction to $r_-(Q^2)$ defined in Eq. (38), which yields

$$r_-(Q^2) = -\frac{4n_f T_R}{3}\sqrt{\frac{2a_s(Q^2)}{C_A}} + O(a_s). \tag{44}$$

For the reader's convenience, we list here expressions with numerical coefficients for $r_+(Q^2)$ through $O(a_s^{3/2})$ and for $r_-(Q^2)$ through $O(\sqrt{a_s})$ in QCD with $n_f = 5$:

$$\begin{aligned} r_+(Q^2) &= 2.25 - 2.18249\sqrt{a_s(Q^2)} - 27.54\,a_s(Q^2) + 10.8462\,a_s^{3/2}(Q^2) + O(a_s^2 \\ r_-(Q^2) &= -2.72166\sqrt{a_s(Q^2)} + O(a_s). \end{aligned} \quad (45$$

In all the approximations considered here, we may summarize our main theoretical results for the avarage gluon and quark jet multiplicities in the following way:

$$\begin{aligned}
\langle n_h(Q^2)\rangle_g &= n_1(Q_0^2)\hat{T}_+^{\mathrm{res}}(0,Q^2,Q_0^2) + n_2(Q_0^2)\, r_-(Q^2)\hat{T}_-^{\mathrm{res}}(0,Q^2,Q_0^2), \\
\langle n_h(Q^2)\rangle_s &= n_1(Q_0^2)\frac{\hat{T}_+^{\mathrm{res}}(0,Q^2,Q_0^2)}{r_+(Q^2)} + n_2(Q_0^2)\, \hat{T}_-^{\mathrm{res}}(0,Q^2,Q_0^2),
\end{aligned} \qquad (46)$$

where

$$\begin{aligned}
n_1(Q_0^2) &= r_+(Q_0^2)\frac{D_g(0,Q_0^2) - r_-(Q_0^2)D_s(0,Q_0^2)}{r_+(Q_0^2) - r_-(Q_0^2)}, \\
n_2(Q_0^2) &= \frac{r_+(Q_0^2)D_s(0,Q_0^2) - D_g(0,Q_0^2)}{r_+(Q_0^2) - r_-(Q_0^2)}.
\end{aligned} \qquad (47)$$

The average gluon-to-quark jet multiplicity ratio may thus be written as

$$r(Q^2) \equiv \frac{\langle n_h(Q^2)\rangle_g}{\langle n_h(Q^2)\rangle_s} = r_+(Q^2)\left[\frac{1 + r_-(Q^2)R(Q_0^2)\frac{\hat{T}_-^{\mathrm{res}}(0,Q^2,Q_0^2)}{\hat{T}_+^{\mathrm{res}}(0,Q^2,Q_0^2)}}{1 + r_+(Q^2)R(Q_0^2)\frac{\hat{T}_-^{\mathrm{res}}(0,Q^2,Q_0^2)}{\hat{T}_+^{\mathrm{res}}(0,Q^2,Q_0^2)}}\right], \tag{48}$$

where

$$R(Q_0^2) = \frac{n_2(Q_0^2)}{n_1(Q_0^2)}. \tag{49}$$

It follows from the definition of $\hat{T}_\pm^{\mathrm{res}}(0, Q^2, Q_0^2$ in Eq. (**??**) and from Eq. (47) that, for $Q^2 = Q_0^2$, Eqs. (46) and (48) become

$$\begin{aligned} &\langle n_h(Q_0^2)\rangle_g = D_g(0, Q_0^2), \quad \langle n_h(Q_0^2)\rangle_q = D_s(0, Q_0^2), \\ &r(Q_0^2) = \frac{D_g(0, Q_0^2)}{D_s(0, Q_0^2)}. \end{aligned} \tag{50}$$

The NNLL-resummed expressions for the average gluon and quark jet multiplicites given by Eq. (46) only depend on two nonperturbative constants, namely $D_g(0, Q_0^2)$ and $D_s(0, Q_0^2)$. These allow for a simple physical interpretation. In fact, according to Eq. (50), they are the average gluon and quark jet multiplicities at the arbitrary scale Q_0.

B7. Analysis

We are now in a position to perform a global fit to the available experimental data of our formulas in Eq. (46) in the $\mathrm{LO}+\mathrm{NNLL}$ ($r_+ = C_A/C_F = 2.25$, $r_- = 0$), $\mathrm{N^3LO_{approx+NNLL}}$ ($r_+ = r_{\mathrm{Capella}}$, $r_- = 0$), , and $\mathrm{N^3LO_{approx+NLO+NNLL}}$ ($r_+ = r_{\mathrm{Capella}}$, $r_- = -2.72166\sqrt{a_s(Q^2)}$) approximations, so as to extract the non-perturbative constants $D_g(0,Q_0^2)$ and $D_s(0,Q_0^2)$.

We have to make a choice for the scale Q_0, which, in principle, is arbitrary. The perturbative series appears to be more rapidly converging at relatively large values of Q_0. Therefore, **we adopt $Q_0 = 50$ GeV in the following.**

	$\mathrm{LO} + \mathrm{NNLL}$	$\mathrm{N^3LO}_{\mathrm{approx+NNLL}}$	$\mathrm{N^3LO}_{\mathrm{approx+NLO+NNLL}}$
$\langle n_h(Q_0^2)\rangle_g$	24.31 ± 0.85	24.02 ± 0.36	24.17 ± 0.36
$\langle n_h(Q_0^2)\rangle_q$	15.49 ± 0.90	15.83 ± 0.37	15.89 ± 0.33
χ^2_{dof}	18.09	3.71	2.92

Table 3: Fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ at $Q_0 = 50$ GeV with 90% CL errors and minimum values of χ^2_{dof} achieved in the $\mathrm{LO} + \mathrm{NNLL}$, $\mathrm{N^3LO}_{\mathrm{approx+NNLL}}$, and $\mathrm{N^3LO}_{\mathrm{approx+NLO+NNLL}}$ approximations.

We included the measurements of average gluon jet multiplicities and those of average quark jet multiplicities, which include 27 and 51 experimental data points, respectively. The results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ at $Q_0 = 50$ GeV together with the χ^2_{dof} values obtained in our $\mathrm{LO} + \mathrm{NNLL}$, $\mathrm{N^3LO}_{\mathrm{approx+NNLL}}$, and $\mathrm{N^3LO}_{\mathrm{approx+NLO+NNLL}}$ fits are listed in Table 3. The errors correspond to 90% CL as explained above. All these fit results are in agreement with the experimental data.

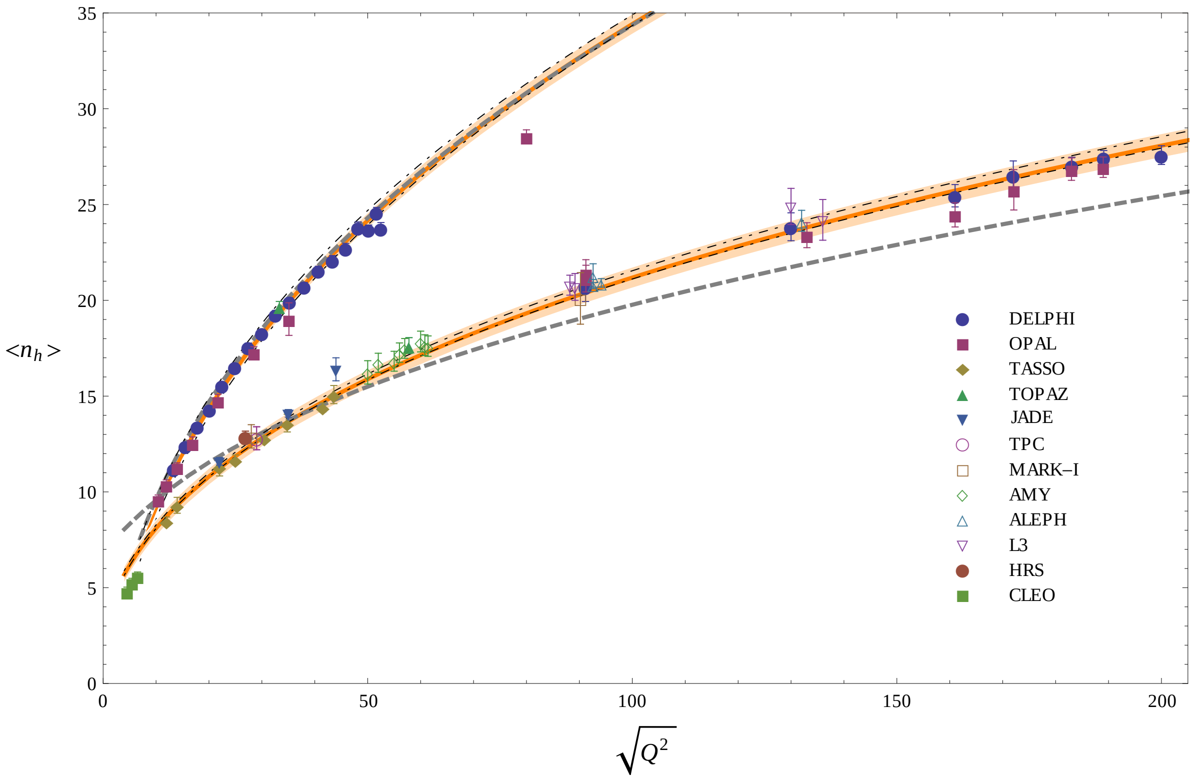


Figure 1: The average gluon (upper curves) and quark (lower curves) jet multiplicities evaluated from Eq. (46), respectively, in the LO + NNLL (dashed/gray lines) and $\mathrm{N^3LO_{approx+NLO+NNLL}}$ (solid/orange lines) approximations using the corresponding fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ from Table 3 are compared with the experimental data included in the fits. The experimental and theoretical uncertainties in the $\mathrm{N^3LO_{approx+NLO+NNLL}}$ results are indicated by the shaded/orange bands and the bands enclosed between the dot-dashed curves, respectively.

In Fig. 1, we show as functions of Q the average gluon and quark jet multiplicities evaluated from Eq. (46) at $\mathrm{LO + NNLL}$ and $\mathrm{N^3LO_{approx+NLO+NNLL}}$ using the corresponding fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ at $Q_0 = 50$ GeV from Table 3.

- Gluon average multiplicity is fitted well like in the previous analysis (A.Capella, I.M.Dremin, J.W.Gary, V.A. Nechitailo and J. Tran Thanh Van, 2000). In a sence, the result (based on the plus components in our approach) should be close to ones obtained in the framework of the famous modified leading-logarithmic approximation (MLLA).

- The fit of quark average multiplicity is good because minus component: there is the additional contribution with the additional free parameter $D_s(0, Q^2)$.

  The quark-singlet minus component comes with an arbitrary normalization and has a slow Q^2 dependence. Consequently, its numerical contribution may be approximately mimicked by a constant introduced to the average quark jet multiplicity as in (P.Abreu et al. [DELPHI Collab.], 1998)

- We can compare our results with the data for the ratio of the gluon and quark average multiplicities. It is not a fit because all our parameters have been already fixed.

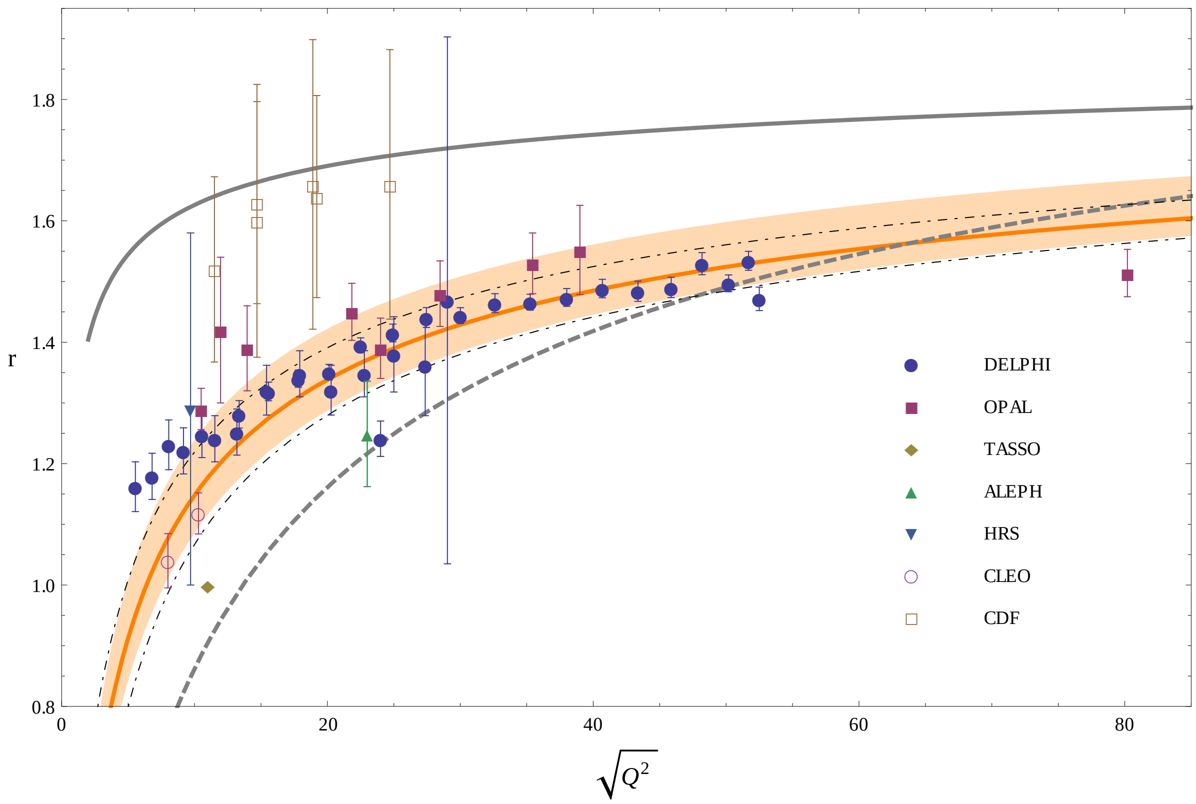


Figure 2: The average gluon-to-quark jet multiplicity ratio evaluated from Eq. (48) in the LO + NNLL (dashe
$\mathrm{N^3LO_{approx+NLO+NNLL}}$ (solid/orange lines) approximations using the corresponding fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ from T
with experimental data. The experimental and theoretical uncertainties in the $\mathrm{N^3LO_{approx+NLO+NNLL}}$ result are indicated by
bands and the bands enclosed between the dot-dashed curves, respectively. The prediction given by analysis in (A.Capella, I.M
V.A. Nechitailo and J. Tran Thanh Van, 2000) is indicated by the continuous/gray line.

B8. Determination of strong-coupling constant

Before we took $\alpha_s^{(5)}(m_Z^2)$ to be a fixed input parameter for our fits. Motivated by the excellent goodness of our $\mathrm{N^3LO_{approx+NNLL}}$ and $\mathrm{N^3LO_{approx+NLO+NNLL}}$ fits, **we now include it among the fit parameters**, the more so as the fits should be sufficiently sensitive to it in view of the wide Q^2 range populated by the experimental data fitted to.

	$\mathrm{N^3LO_{approx+NNLL}}$	$\mathrm{N^3LO_{approx+NLO+NNLL}}$
$\langle n_h(Q_0^2)\rangle_g$	24.18 ± 0.32	24.22 ± 0.33
$\langle n_h(Q_0^2)\rangle_q$	15.86 ± 0.37	15.88 ± 0.35
$\alpha_s^{(5)}(m_Z^2)$	0.1242 ± 0.0046	0.1199 ± 0.0044
χ^2_{dof}	2.84	2.85

Table 4: Fit results for $\langle n_h(Q_0^2)\rangle_g$ and $\langle n_h(Q_0^2)\rangle_q$ at $Q_0 = 50$ GeV and for $\alpha_s^{(5)}(m_Z^2)$ with 90% CL errors and minimum values of χ^2_{dof} achieved in the $\mathrm{N^3LO_{approx+NNLL}}$ and $\mathrm{N^3LO_{approx+NLO+NNLL}}$ approximations.

We fit to the same experimental data as before and again put $Q_0 = 50$ GeV. The fit results are summarized in Table 4.

We observe from Table 4 that the results of the $\mathrm{N^3LO_{approx+NNLL}}$ **and** $\mathrm{N^3LO_{approx+NLO+NNLL}}$ **fits for** $\langle n_h(Q_0^2)\rangle_g$ **and** $\langle n_h(Q_0^2)\rangle_q$ **are mutually consistent. They are also consistent with the respective fit results in Table 3.** As expected, the values of χ^2_{dof} are reduced by relasing $\alpha_s^{(5)}(m_Z^2)$ in the fits, from 3.71 to 2.84 in the $\mathrm{N^3LO_{approx+NNLL}}$ approximation and from 2.95 to 2.85 in the $\mathrm{N^3LO_{approx+NLO+NNLL}}$ one. The three-parameter fits strongly confine $\alpha_s^{(5)}(m_Z^2)$, within an error of 3.7% at 90% CL in both approximations. **The inclusion of the** $r_-(Q^2)$ **term has the beneficial effect of shifting** $\alpha_s^{(5)}(m_Z^2)$ **closer to the world average,** 0.1184 ± 0.0007 **(J.Beringer et al. [Particle Data Group Collab.], 2012)**

B9. Conclusion II

- Prior to our analysis, experimental data on the average gluon and quark jet multiplicities could not be simultaneously described in a satisfactory way mainly because the theoretical formalism failed to account for the difference in hadronic contents between gluon and quark jets, although the convergence of perturbation theory seemed to be well under control.

- This problem may be solved by including the minus components governed by $\hat{T}_{-}^{\mathrm{res}}(0, Q^2, Q_0^2)$ The quark-singlet minus component comes with an arbitrary normalization and has a slow Q^2 dependence. Consequently, its numerical contribution may be approximately mimicked by a constant introduced to the average quark jet multiplicity as in (P.Abreu et al. [DELPHI Collab.], 1998).

- Motivated by the goodness of our $\mathrm{N^3LO_{approx+NNLL}}$ and $\mathrm{N^3LO_{approx+NLO+N}}$ fits with fixed value of $\alpha_s^{(5)}(m_Z^2)$ here, we then included $\alpha_s^{(5)}(m_Z^2)$ among the fit parameters, which yielded a further reduction of χ^2_{dof}. The obtained value $\alpha_s^{(5)}(m_Z^2) = 0.1199 \pm 0.0026$ is close to the world average one.

Next steps:

- We have some problems with a proper resummation of the non-diagonal terms in the Q^2-evolution. We hope to improve this part of our analysis in our future studies.
- To study the fragmentation functions themselves.