Six-loop calculations of the critical exponents in the ϕ^4 theory

Dmitrii Batkovich[1], Konstantin Chetyrkin[2], Mikhail Kompaniets[1]

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Outline

- Motivation
- R'-operation, *IRR* (Infrared Rearrangement), *IBP* reduction and R^*-operation
- Tools (GraphState, Graphine, IBP Reduction, R^*)
- Results, discussion

$O(n)$-symmetric ϕ^4 model

$$S(\varphi) = -\int d\mathbf{x} \left(\frac{1}{2}\tau\varphi(\mathbf{x})^2 + \frac{1}{2}(\overrightarrow{\bigtriangledown}\varphi(\mathbf{x}))^2 + \frac{1}{24}g(\varphi(\mathbf{x})^2)^2 \right)$$

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- This work: critical exponent η at 6-loop level

Bogoliubov–Parasyuk R' operation and infrared rearrangement

Bogoliubov–Parasyuk R-operation (and incomplete R'-operation) plays significant role in multiloop calculations.

$$\delta Z = \Delta_{UV}(\gamma) = -KR'(\gamma), \quad R'(\gamma) = \gamma + \sum_{\delta \text{ is UV}} \Delta_{UV}(\delta)\gamma/\delta$$

where K is subtraction operator, for MS-scheme: $K\left(\sum_{k=-\infty}^{\infty} a_k \varepsilon^k\right) = \sum_{k=-\infty}^{-1} a_k \varepsilon^k$
and R'-operation subtracts all *proper* UV subdiveregences.
counterterm $\Delta_{UV}(\Gamma) = -KR'(\Gamma)$ of the graph Γ is **polynomial on masses and external momentums**[1].

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For logarithmically divergent graphs this allows one to perform **infrared rearrangement**:

- to set some masses and external moments to zero
- or to introduce new external momenta or masses

All these operations can be done if they are does not produce unphysical **infrared divergences** and result doesn't depend on procedure details.[2]

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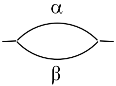
G-functions and integration by parts reduction

Limited class of massless diagrams with one external momenta can be evaluated analytically using

$$\frac{1}{(2\pi)^d}\int dk \frac{1}{k^{2\alpha}(k-p)^{2\beta}} = \frac{1}{p^{2(\alpha+\beta-\lambda-1)}}G(\alpha,\beta)$$

where $\lambda = d/2-1$, $G(\alpha,\beta) = \frac{\Gamma(\lambda+1-\alpha)\Gamma(\lambda+1-\beta)\Gamma(\alpha+\beta-\lambda-1)}{(4\pi)^{\lambda+1}\Gamma(\alpha)\Gamma(\beta)\Gamma(2\lambda+2-\alpha-\beta)}$



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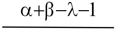

$G(\alpha,\beta) = \frac{1}{p^{2(\alpha+\beta-\lambda-1)}}G(\alpha,\beta)$

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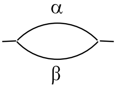

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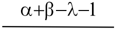

$$G(\alpha,\beta) = \frac{1}{p^{2(\alpha+\beta-\lambda-1)}} G(\alpha,\beta)$$

One of the most powerful analytical method for evaluation of diagrams depending on one external momentum (p-integral) is IBP (Integration by Parts) reduction. **IBP reduction** is the set of rules to express diagram value through the values of some predefined diagrams, calls master integrals. This procedure can be fully **automated**

We use **LiteRed** program for generating IBP and DRR[3] rules for reduction[4]. All 4-loop masters we need are available from[5] and from[6].

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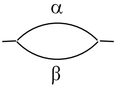

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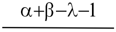

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But with 4-loop *IBP* reduction + G-function reduction and R' + *IRR* can be calculated **only 18** *UV*-counterterms of 6−loop propagator diagrams **from 50** in ϕ^4.

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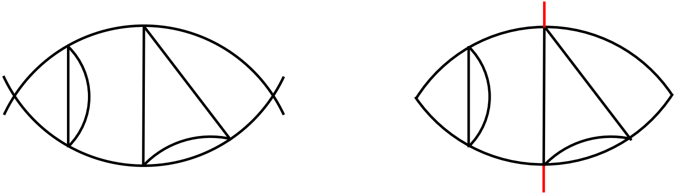
Overcoming of *IRR* restrictions using R^*



No way to calculate left diagram using R' and 4-loop *IBP* reduction.

Right diagram calculable with G-functions (no reduction required!) but with such rearrangement IR divergence arise, and to get correct result we need to use $R^{*\prime}$ operation to subtract this divergences.

R^*-operation historical overview

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- We employ a version of R^* operation as developed in **K.G. Chetyrkin**, *MPI-Ph/PTh*, 13/91, 1991

$R^{*\prime}$ definition

$$R^{*\prime}(\Gamma) = \widetilde{R}' R'(\Gamma)$$

where R' is incomplete R-operation removing divergences in subgraphs. $\widetilde{R}'$ is IR-counterpart of R', it removes $IR-$divergences arising at choosing the wrong momentum rearrangement in IRR point of view.

$$\begin{aligned} \widetilde{R}'(\Gamma) &= \Gamma + \sum_{\gamma \in W_u(\Gamma)} \Delta_{IR}(\Gamma/\gamma)\gamma, \text{ if } \Gamma \text{ is not 0-tadpole} \\ \widetilde{R}'(\Gamma) &= \Delta_{IR}(\Gamma), \text{ if } \Gamma \text{ is 0-tadpole,} \end{aligned}$$

Here **by 0-tadpole we denote massless tadpole**, $W_u(\Gamma)$ - set of all UV-divergent subgraphs of Γ through which external momentum of Γ can be completely passed and subgraph doesn't contain tadpoles.

$$\boxed{\Delta_{UV}(\Gamma) = -KR^{*\prime}(\Gamma)}$$

Recursive calculation of IR-counterterms

Let Γ is 0-tadpole:

$$\Delta_{UV}(\Gamma) = -K\widetilde{R}'R'(\Gamma) = -K\Delta_{IR}R'(\Gamma)$$

$$\boxed{\Delta_{UV}(\Gamma) + \Delta_{IR}R'(\Gamma) = 0}$$

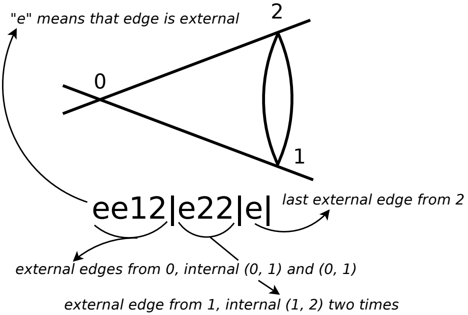
Tools[7][8]

- GraphState
  - graph serialization to compact easy-readable form and its deserialization
  - graph isomorphisms searching
  - symmetry factors
- Graphine
  - graph-theoretical manupulations
  - subgraph searching for given condition
- IBP reduction
  - reduction up to 4-loop using **LiteRed** rules
- R^* (not published yet)
  - calculation of UV counterterms using R' and $R^{*\prime}$ operations
  - G-functions reduction
  - custom graph calculators

[7] https://code.google.com/p/rg-graph/

[8] Written on Python

GraphState: Nickel notation for undirected graphs



- each letter (excluding '|') represents one edge
- edges separated by '|' on groups where all edges have equal minimal vertex index ('e' $<$ any number)
- edges groups ordered by vertex index order
- minimal notation in lexigraphical order sense chosen as canonical

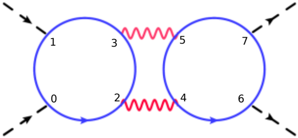
GraphState: Nickel notation generalization

Generalization:

- custom directed and undirected edges properties
- custom vertex properties

One can identify and manipulate with graphs for complicated theories like Standard Model:

Higgs self-coupling in the Standad Model[9]



$HH^{+} \to hH$

$gg \to GG$

$q\bar{q} \to qQ$

e12|e3|34|5|56|7|e7|e| : 0H_Qq_qQ|0H_Qq|qQ_GG|GG|Qq_qQ|Qa|0h_qQ|0h|

```
import graph_state

fields_key = graph_state.PropertyKey(name="fields", is_edge_property=True, is_directed=True,
        externalizer=graph_state.Fields.externalizer())

edges_with_fields_config = graph_state.PropertiesConfig.create(fields_key)

bubble_with_fields = edges_with_fields_config.graph_state_from_str("e11|e|:0h_qQ_Qq|0H|")
print str(bubble_with_fields) # "e11|e|:0H_Qq_qQ|0h|"
```

[9] **A. V. Bednyakov, A. F. Pikelner and V. N. Velizhanin**, Nucl.Phys. **B879** (2014) 256-267

Graphine: Relevant subgraphs searching

Graphine:

- graph-theoretical manupulations (connectivity components, subgraphs shrinking, etc.)
- subgraph searching for given condition

Graphine allows to find relevant subgraphs of initial graph relevant to given condition (filter). Condition can be defined as a function:

```
import graphine

@graphine.filters.graph_filter
def has_proper_uv_index(graph_edges, super_graph):
   # ..do some boolean decision
    graph = graphine.Graph(edges_list)
    uv_index = 4 * graph.loops_count - 2 * graph.internal_edges_count
    return uv_index >= 0
```

Now using filter above anf some predefined filters we can find UV-divergent subgraphs as follows:

```
from graphine.filters import one_irreducible, no_tadpole_in_cograph

for uv_sg in graph.x_relevant_sub_graphs(filters=one_irreducible + has_proper_uv_index + no_tadpole_in_cograph):
    print uv_sg
```

Implementation of R' operation using Graphine

Sample implementation of R'-operation:

- has_proper_uv_index(...) – filter for identifying UV-divergent subgraphs
- val(graph) – some function which calculates graph value or takes it form database
- k(value) – implementation for K operation (e.g. MS scheme)

```
import graphine
from graphine.filters import one_irreducible, graph_filter

@graph_filter
def has_proper_uv_index(edges_list, super_graph):
    graph = graphine.Graph(edges_list)
    uv_index = 4 * graph.loops_count - 2 * graph.internal_edges_count
    return uv_index >= 0

def val(graph):
    ...

def k(value):
    ...

def r1(graph):
    _r1 = val(graph)
    uv_subgraphs = [uv_sg for uv_sg in graph.x_relevant_sub_graphs(filters=one_irreducible + has_proper_uv_index +
            no_tadpole_in_cograph)]

    for i in xrange(1, len(uv_subgraphs) + 1):
        for uv_comb in itertools.combinations(uv_subgraphs, i):
            if i == 1 or not graphine.util.has_intersecting_by_vertices_graphs(uv_comb):
                co_sg = graph.shrink_to_point(uv_comb)
                _r1 += reduce(lambda v, g: v * k(r1(g)), uv_comb, 1) * val(co_sg)

    return _r1
```

Results

Six-loop diagrams overview

6-loop diagrams of propagator and vertex type 1-PI Green functions:

	total	factorized	primitive	4-loop reducible	4-loop irreducible
Γ_2	50	0	0	48	2
Γ_4	627	124	10	481	12

Γ_2 have been calculated. Γ_4 in progress now.

48 of 50 Γ_2-diagrams can be easily calculated using $R^{*\prime}$-operation and 4-loop IBP reduction, the last 2 requires some additional tricks

Results

Six-loops critical exponent η expansion, $N = 1$:

$$\begin{aligned}
\eta &= \frac{2}{27}\varepsilon^2 + \frac{109}{729}\varepsilon^3 + \left(\frac{7217}{39366} - \frac{64}{243}\zeta_3\right)\varepsilon^4 \\
&+ \left(\frac{321511}{2125764} - \frac{16}{3645}\pi^4 - \frac{1316}{2187}\zeta_3 + \frac{1280}{729}\zeta_5\right)\varepsilon^5 \\
&+ \left(\frac{3421613}{38263752} - \frac{3136}{243}\zeta_7 + \frac{73232}{19683}\zeta_5 - \frac{181462}{177147}\zeta_3 + \frac{640}{137781}\pi^6\right. \\
&+ \left.\frac{2432}{2187}\zeta_3^2 - \frac{329}{32805}\pi^4\right)\varepsilon^6 + \mathcal{O}(\varepsilon^7)
\end{aligned}$$

Result verification

- each graph have calculated by all possible ways using R', $R^{*\prime}$ with all possible rearrangements and different levels of reduction

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- each graph have calculated by all possible ways using R', $R^{*\prime}$ with all possible rearrangements and different levels of reduction
- there is no $log(p)$ in Δ_{UV} and Δ_{IR}
- some of diagrams calculated numerically with Sector Decomposition
- checking through the known $1/N$-expansion of η (up to $1/N^3$ for arbitrary ε)

$1/N$-expansion critical exponent η

Vasiliev, Pis'mak and Honkonen, *Theor. Math. Phys.*, 50,N 2, p127 (1982)[10]:

$$\eta_N = \frac{\eta_1}{N} + \frac{\eta_2}{N^2} + \frac{\eta_3}{N^3} + \mathcal{O}\left(\frac{1}{N^4}\right) \quad \text{for any } \varepsilon,$$

We can expand η_N by ε. But we know η_ε up to 6-loops for any N.

$$\boxed{\eta_N \text{ expansion by } \varepsilon = \eta_\varepsilon \text{ expansion by } 1/N}$$

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All of 6-loops self-energy diagrams Δ_{UV} give contribution to η_3.

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All of 6-loops self-energy diagrams Δ_{UV} give contribution to η_3.

We have indeed found FULL agreement to this 32 years old prediction!

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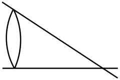
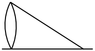
Thank you for your attention!

Backup

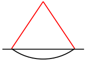
IR-divergences



Two momentum rearrangements:



$$\sim \int \frac{1}{k_1^2} \frac{1}{(q-k_1)^2} \frac{1}{k_2^2} \frac{1}{(k_1-k_2)^2} dk$$

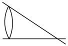
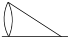
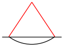


$$\sim \int \frac{1}{k_1^2} \left(\frac{1}{k_2^2}\right)^2 \frac{1}{(q-k_1-k_2)^2} dk$$

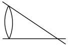
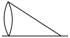
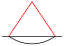
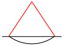
Asymptotic of second integrand for $k_2 \sim 0$:

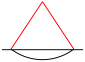
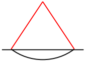
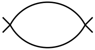
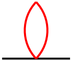
$$\frac{dk_2}{k_2^4} \sim \frac{|k_2|^{3-2\varepsilon} d|k_2|}{|k_2|^4} = |k_2|^{-1-2\varepsilon} d|k_2|$$

Example 0

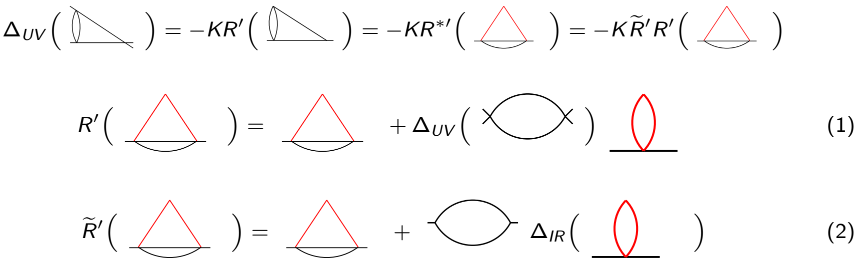
$\Delta_{UV}\Big($  $\Big) = -KR'\Big($  $\Big) = -KR^{*\prime}\Big($  $\Big) = -K\widetilde{R}'R'\Big($  $\Big)$

Example 0

$\Delta_{UV}\Big($  $\Big) = -KR'\Big($  $\Big) = -KR^{*\prime}\Big($  $\Big) = -K\widetilde{R}'R'\Big($  $\Big)$

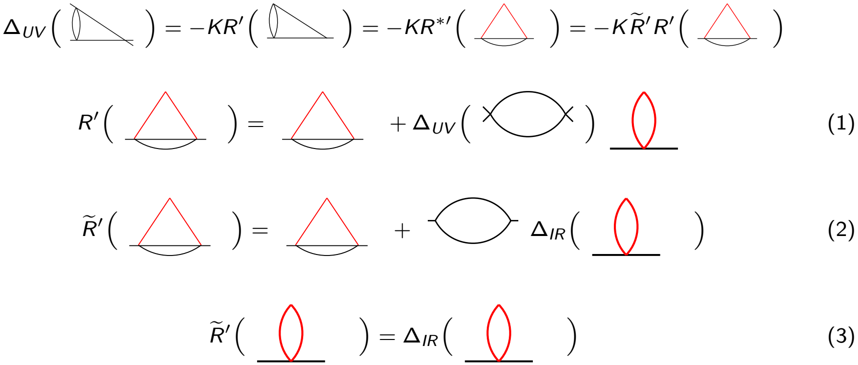
$R'\Big($  $\Big) =$  $+\,\Delta_{UV}\Big($  $\Big)$  (1)

Example 0

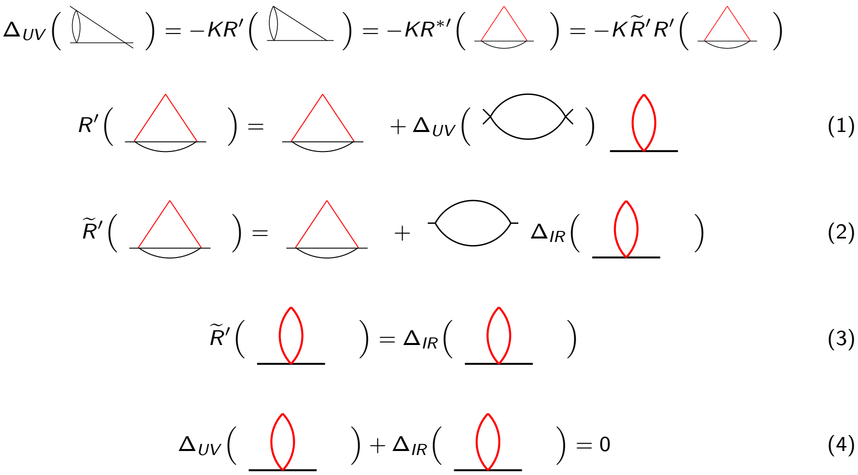


Δ_{UV} ( ) $= -KR'$ ( ) $= -KR^{*\prime}$ ( ) $= -K\widetilde{R}'R'$ ( )
R' ( ) $=$ $+ \Delta_{UV}$ ( )
(1)
$\widetilde{R}'$ ( ) $=$ $+$ Δ_{IR} ( )
(2)

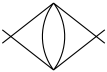
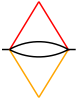
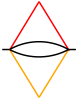
Example 0



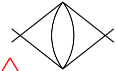
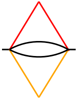
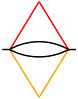
Example 0

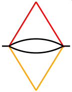
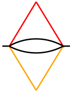
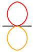


Example 1

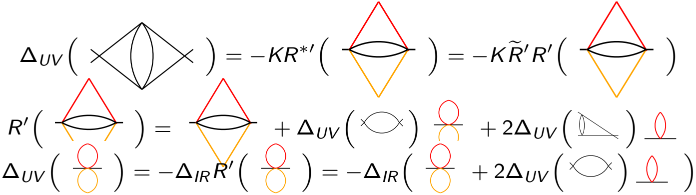
$\Delta_{UV}\Big($  $\Big) = -KR^{*\prime}\Big($  $\Big) = -K\widetilde{R}'R'\Big($  $\Big)$

Example 1

$\Delta_{UV}\Big($  $\Big) = -KR^{*\prime}\Big($  $\Big) = -K\widetilde{R}'R'\Big($  $\Big)$

$R'\Big($  $\Big) =$  $+\Delta_{UV}\Big($  $\Big)$  $+2\Delta_{UV}\Big($  $\Big)$  (5)

Example 1

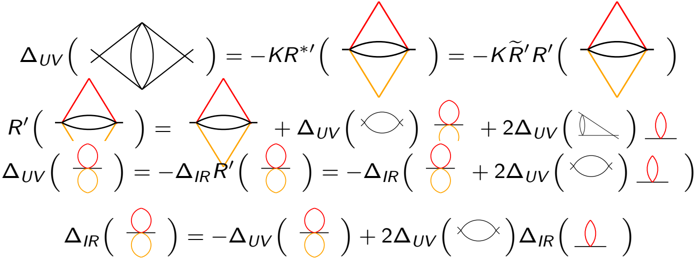


Δ_{UV} ( ) $= -KR^{*\prime}$ ( ) $= -K\widetilde{R}' R'$ ( )
R' ( ) $=$ $+ \Delta_{UV}$ ( ) $+ 2\Delta_{UV}$ ( )
Δ_{UV} ( ) $= -\Delta_{IR} R'$ ( ) $= -\Delta_{IR}$ ($+ 2\Delta_{UV}$ ( ))

(5)

(6)

Example 1

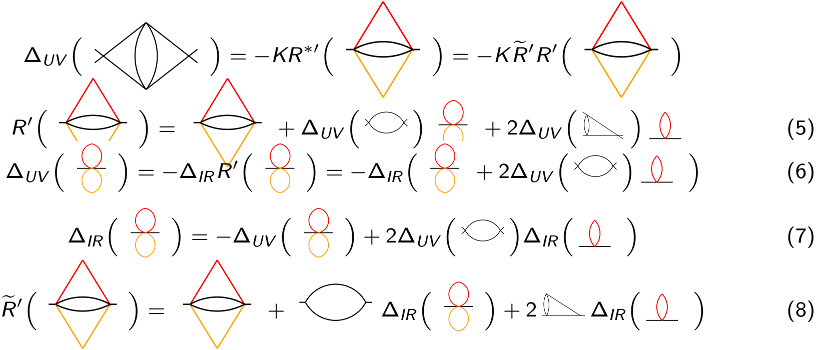


(5)

(6)

(7)

Example 1



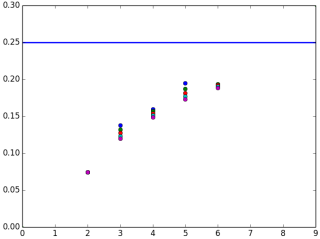
Δ_UV ( ) = −KR*′ ( ) = −KR̃′R′ ( )
R′ ( ) = + Δ_UV ( ) + 2Δ_UV ( ) (5)
Δ_UV ( ) = −Δ_IR R′ ( ) = −Δ_IR (+ 2Δ_UV ( )) (6)
Δ_IR ( ) = −Δ_UV ( ) + 2Δ_UV ( ) Δ_IR ( ) (7)
R̃′ ( ) = + Δ_IR ( ) + 2 Δ_IR ( ) (8)

Results

Numerical six-loop critical exponent η expansion, $N = 1$:

$$\eta = 0.0740\varepsilon^2 + 0.1495\varepsilon^3 - 0.1332\varepsilon^4 + 0.8210\varepsilon^5 - 5.2014\varepsilon^6 + \mathcal{O}(\varepsilon^7)$$

Resummation using Borel transform and conformal mapping for $d = 2$, $N = 1$



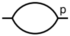
Recursive calculation of *IR*-counterterms

Let Γ is 0-tadpole:

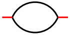
$$\Delta_{UV}(\Gamma) = -K\widetilde{R}'R'(\Gamma) = -K\Delta_{IR}R'(\Gamma)$$

$$\boxed{\Delta_{UV}(\Gamma) + \Delta_{IR}R'(\Gamma) = 0}$$

Physical sense:

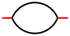
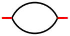


has no IR, only UV divergences

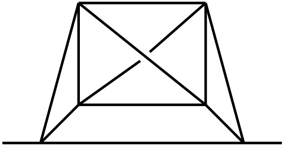


has IR and UV divergences but diagram identically equals zero

Hence

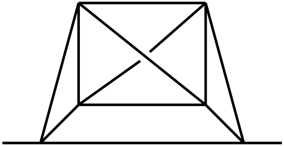
$\Delta_{UV}\Big($  $\Big) + \Delta_{IR}\Big($  $\Big) = 0$

4-loop irreducible self-energy diagrams

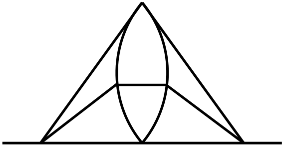


Calculated using **reconstruction of pole part** of the 5-loop graph from known UV-counterterm.

4-loop irreducible self-energy diagrams



Calculated using **reconstruction of pole part** of the 5-loop graph from known UV-counterterm.



Calculated using **dual graphs**.

Calculation of graph pole part

Proposition

Consider graph Γ with $N-$loops, then its value up to $\mathcal{O}(\varepsilon^0)$ can be expressed through

- $\Delta_{UV}(\Gamma)$
- Δ_{UV} *and values of diagrams with loops count less then N.*

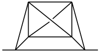
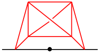
Proof.

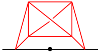
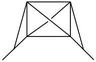
$$\Delta_{UV}(\Gamma) = -KR^{*\prime}(\Gamma) = -K(\Gamma) + (K(\Gamma) - KR^{*\prime}(\Gamma)) = -K(\Gamma) + \widehat{\Delta}_{UV}(\Gamma)$$

$$\Gamma = K(\Gamma) + \mathcal{O}(\varepsilon^0) = -\Delta_{UV}(\Gamma) + \widehat{\Delta}_{UV}(\Gamma) + \mathcal{O}(\varepsilon^0)$$

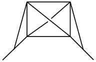
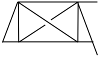
□

Calculation of graph pole part

$\Delta_{UV}\Big(\frac{1}{2}\partial^2 p$  $\Big) = \Delta_{UV}\Big(\frac{4-d}{d}$  $\Big)$

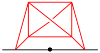
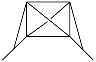
$K\frac{4-d}{d}$  $= K\frac{-\varepsilon}{2-\varepsilon}G(2, 5-5\lambda)$ 

$$\frac{-\varepsilon}{2-\varepsilon}G(2, 5-5\lambda) \sim \mathrm{Const} + \mathcal{O}(\varepsilon),$$

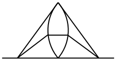
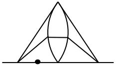
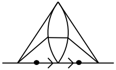
$\Delta_{UV}\Big($  $\Big) = -KR^{*\prime}\Big($  $\Big)$

where $\lambda = d/2 - 1$.

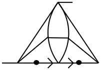
Hence

$K\frac{4-d}{d}$  $= K\frac{-\varepsilon}{2-\varepsilon}G(2, 5-5\lambda)K\Big($  $\Big)$

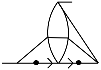
Calculation via dual diagram

$$\Delta_{UV}\left(\frac{1}{2}\partial^2 p$$  $$\right) = \Delta_{UV}\left(2\frac{4-d}{d}$$  $$\right) + \Delta_{UV}\left(2\frac{4}{d}$$  $$\right)$$

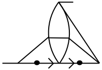
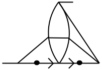
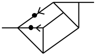
For 2nd contribution it is possible to rearrange external momenta in the following way:



Now to get the value we need to calculate this (primitive) graph:



It is possible to calculate this graph by the following well known trick
that graph in x-space is equal to dual graph in p-space

$\Big($  $\Big)_{\text{p-space}} = C \cdot \Big($  $\Big)_{\text{x-space}} = C \cdot \Big($  $\Big)_{\text{p-space}}$

Additional simplification comes from the fact that this graph is primitive
and we need only leading term from this graph so we can put all line indices equal to 1
and calculate this graph using 4-loop reduction