Mixing of fermions and spectral representation of propagator

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Introduction

Effects of mixing in systems of neutrinos and quarks are studied now intensively, both experimentally and theoretically. General tendency now is the passing to the quantum field methods instead of simplified quantum-mechanical description. So, the main objects becomes the dressed propagators and vertices.

We discuss here the QFT approach to mixing problem, the main object in our consideration is the matrix propagator. But this traditional issue is considered in the non-standard frameworks. We will obtain main equations of this approach and discuss their properties.

Possible applications — mixing of neutinos or quarks.

Spectral representation of operator

We use for our purposes the so called **spectral representation** of operator. Recall it with use of quantum-mechanical notations.
Let us consider the eigenstate problem for some self-adjont operator $\hat{A}$

$$\hat{A}|i\rangle = \lambda_i|i\rangle.$$

This operator may be represented as

$$\hat{A} = \sum_i \lambda_i \Pi_i = \sum_i \lambda_i |i\rangle\langle i|,$$

and $\Pi_i = |i\rangle\langle i|$ are corresponding projectors (eigenprojectors).
Completeness condition (decomposition of unit):

$$1 = \sum_i \Pi_i = \sum_i |i\rangle\langle i|.$$

The similar representation exists for non self-adjont operator, but one needs to consider both left and right eigenstate problems.

Eigenstate problem for fermion propagator

We consider the eigenstate problem for inverse fermion propagator S

$$S|i\rangle = \lambda_i |i\rangle.$$

It's more convenient to solve this problem for eigenprojectors $\Pi_i = |i\rangle\langle i|$

$$S\Pi_i = \lambda_i \Pi_i.$$

If Π_i is a complete set of orthogonal projectors, we can represent S in the form

$$S = \sum_i \lambda_i \Pi_i,$$

and propagator G looks as

$$G = \sum_i \frac{1}{\lambda_i} \Pi_i.$$

Poles of propagator = zeros of eigenvalues

Single fermion (parity is conserved)

The solution in simple cases — the known **off-shell** projectors

$$\Lambda^{\pm} = \frac{1}{2}\left(1 \pm \frac{\hat{p}}{W}\right), \tag{1}$$

where $W = \sqrt{p^2}$ is the rest-frame energy.

For bare inverse propagator:

$$\begin{aligned} \Pi_1 &= \Lambda^+, \qquad \lambda_1 = (W - m), & (2)\\ \Pi_2 &= \Lambda^-, \qquad \lambda_2 = (-W - m). & (3) \end{aligned}$$

Spectral representation:

$$S_0 = \hat{p} - m = \Lambda^+(W - m) + \Lambda^-(-W - m) \tag{4}$$

Bare propagator looks as

$$G_0 = \frac{1}{\hat{p} - m} = \Lambda^+ \frac{1}{W - m} + \Lambda^- \frac{1}{-W - m}. \tag{5}$$

Single fermion (parity is conserved)

The dressing of propagator also looks very simple

$$G_0 \Rightarrow \quad G = \Lambda^+ \frac{1}{W - m - \Sigma_1(W)} + \Lambda^- \frac{1}{-W - m - \Sigma_2(W)}, \qquad (6)$$

where the self-energy also is decomposed in this basis

$$\Sigma(p) = A(p^2) + \hat{p}B(p^2) = \Lambda^+(A + WB) + \Lambda^-(A - WB) \equiv$$
$$\equiv \Lambda^+\Sigma_1(W) + \Lambda^-\Sigma_2(W).$$

Components of self-energy (by construction) are related with each other

$$\Sigma_2(W) = \Sigma_1(-W)$$

and the same is true for eigenvalues

$$\lambda_2(W) = \lambda_1(-W).$$

Single fermion in case of P-parity violation

It gives the first non-trivial example.
In case of parity violation the projection basis must be supplemented by elements with γ^5, it is handy to choose the basis as

$$\mathcal{P}_1 = \Lambda^+, \quad \mathcal{P}_2 = \Lambda^-, \quad \mathcal{P}_3 = \Lambda^+\gamma^5, \quad \mathcal{P}_4 = \Lambda^+\gamma^5. \tag{7}$$

Now the decomposition of a self-energy or a propagator has four terms

$$S = \sum_{M=1}^{4} S_M \mathcal{P}_M, \tag{8}$$

where coefficients S_M are followed by obvious symmetry properties

$$S_2(W) = S_1(-W), \quad S_4(W) = S_3(-W) \tag{9}$$

We need to solve the eigenstate problem for inverse propagator $S(p)$

$$S\Pi_k = \lambda_k \Pi_k.$$

Single fermion in case of P-parity violation

The problem can be solved in the most general case. Let $S(p)$ is defined by decomposition (8) with arbitrary coefficients, the matrix Π also can be written in such form with some coefficients a_M.
Characteristic equation for λ_i:

$$\lambda^2 - \lambda(S_1 + S_2) + (S_1 S_2 - S_3 S_4) = 0, \tag{10}$$

After some algebra we can obtain the projectors

$$\begin{aligned} \Pi_1 &= \frac{1}{\lambda_2 - \lambda_1}\Big((S_2 - \lambda_1)\mathcal{P}_1 + (S_1 - \lambda_1)\mathcal{P}_2 - S_3\mathcal{P}_3 - S_4\mathcal{P}_4\Big), \\ \Pi_2 &= \frac{1}{\lambda_1 - \lambda_2}\Big((S_2 - \lambda_2)\mathcal{P}_1 + (S_1 - \lambda_2)\mathcal{P}_2 - S_3\mathcal{P}_3 - S_4\mathcal{P}_4\Big). \end{aligned} \tag{11}$$

with desired properties:

- $S\Pi_k = \lambda_k \Pi_k$, where an eigenvalue λ_k is a root of equation (10),
- $\Pi_k^2 = \Pi_k$,
- $\Pi_1\Pi_2 = \Pi_2\Pi_1 = 0$,
- $\Pi_1 + \Pi_2 = 1$.

Single fermion in case of P-parity violation

Spectral representation allows easy to perform the renormalization of fermion propagator both in OMS and pole schemes of renormalization.

Details: A.E. Kaloshin, V.P. Lomov. Top quark as a resonance, Eur.Phys.J. C72 (2012) 2094

Two small comments.

- Form of resonance in case of parity violation looks rather non-standard. Resonance factor $1/\lambda_1$ in vicinity of $W = m$ after OMS renormalization looks as

$$\frac{1}{\lambda_1(W)} \approx \frac{1}{W\sqrt{1+\imath\frac{\Gamma(W)}{KW}} - m} \simeq \frac{1}{W - m + \imath\Gamma(W)/2} \qquad (12)$$

  and only at small Γ resonance factor returns to standard form. **(! γ^5 matrix takes part in Dyson summation)**
- Spin projectors do not commutate with dressed propagator in case of parity violation.

Mixing of fermions

Let the inverse propagator is defined by decomposition

$$S = \sum_{M=1}^{4} \mathcal{P}_M S_M,$$

where the coefficients S_M are the known matrices $n \times n$.
As for $n = 1$ case, we prefer to solve the eigenstate problem

$$S\Pi = \lambda\Pi \tag{13}$$

in matrix form, i.e. we are looking for eigenprojectors Π instead of eigenvectors. The desired eigenprojectors also can be written as such decomposition

$$\Pi = \sum_{M=1}^{4} \mathcal{P}_M A_M, \tag{14}$$

with matrix $n \times n$ coefficients A_M.

Mixing of fermions

Recall notations for basis

$$\mathcal{P}_1 = \Lambda^+, \quad \mathcal{P}_2 = \Lambda^-, \quad \mathcal{P}_3 = \Lambda^+\gamma^5, \quad \mathcal{P}_4 = \Lambda^-\gamma^5,$$
$$\Lambda^{\pm}(p) = \frac{1}{2}\Big(1 \pm \frac{\hat{p}}{W}\Big), \qquad W = \sqrt{p^2}. \tag{15}$$

Recall, that S (and Π also) has two sets of indexes $S_{\alpha\beta;ab}$, where $\alpha, \beta = 1, \ldots, 4$ are the Dirac γ-matrix indexes and $a, b = 1, \ldots, n$.

Due to simple multiplicative properties of basis, the eigenstate problem gives set of matrix equations

$$\begin{aligned}
(S_1 - \lambda)A_1 + S_3 A_4 &= 0 \\
(S_2 - \lambda)A_2 + S_4 A_3 &= 0 \\
(S_1 - \lambda)A_3 + S_3 A_2 &= 0 \\
(S_2 - \lambda)A_4 + S_4 A_1 &= 0
\end{aligned} \tag{16}$$

Mixing of fermions

As a result, the $n \times n$ matrices A_1, A_2 should satisfy the homogeneous equations

$$\begin{aligned} \hat{O} A_1 &\equiv [(S_2 - \lambda) S_3^{-1} (S_1 - \lambda) - S_4] A_1 = 0, \\ \hat{O}' A_2 &\equiv [(S_1 - \lambda) S_4^{-1} (S_2 - \lambda) - S_3] A_2 = 0 \end{aligned} \tag{17}$$

and A_3, A_4 are related with them by

$$A_3 = -S_4^{-1} (S_2 - \lambda) A_2, \quad A_4 = -S_3^{-1} (S_1 - \lambda) A_1. \tag{18}$$

One can see that matrices $\hat{O}$, $\hat{O}'$ are related with each other by

$$\hat{O}' = (S_1 - \lambda) S_4^{-1} \cdot \hat{O} \cdot (S_1 - \lambda)^{-1} S_3, \tag{19}$$

so equations (17) give the same characteristic equation

$$\det[(S_2 - \lambda) S_3^{-1} (S_1 - \lambda) - S_4] = 0. \tag{20}$$

In the absence of degeneration this equation gives $2n$ different eigenvalues $\lambda_i(W)$.

Mixing of fermions

As a result the matrix solution of left eigenstate problem takes the form

$$\Pi^i = \mathcal{P}_1 A_1^i + \mathcal{P}_2 A_2^i - \mathcal{P}_3 S_4^{-1}(S_2 - \lambda_i) A_2^i - \mathcal{P}_4 S_3^{-1}(S_1 - \lambda_i) A_1^i, \quad (21)$$

where A_1^i, A_2^i are solutions of equations

$$\begin{aligned} \hat{O}_i A_1^i \equiv \hat{O}(\lambda = \lambda_i) A_1^i = 0, \\ \hat{O}'_i A_2^i \equiv \hat{O}'(\lambda = \lambda_i) A_2^i = 0 \end{aligned} \quad (22)$$

and eigenvalues $\lambda_i(W)$ are defined by equation (20).

Right eigenstate problem

As the next step consider the right eigenstate problem

$$\Pi_R S = \lambda \Pi_R. \tag{23}$$

We can look for the right eigenprojectors Π_R in the same form (14) with matrix coefficients B_M. Similar calculations give the matrix solution of the right problem

$$\Pi_R^i = \mathcal{P}_1 B_1^i + \mathcal{P}_2 B_2^i - \mathcal{P}_3 B_1^i S_3 (S_2 - \lambda_i)^{-1} - \mathcal{P}_4 B_2^i S_4 (S_1 - \lambda_i)^{-1}, \tag{24}$$

where B_1^i, B_2^i are solutions of the left homogeneous equations

$$B_1^i \hat{O}_i' = 0, \qquad B_2^i \hat{O}_i = 0 \tag{25}$$

and eigenvalues $\lambda_i(W)$ are defined by the same equation (20).

Left and right eigenstate problems together

Let us require matrix Π to be a solution of both left and right eigenstate problems.

First of all, $B_1^i = A_1^i$, $B_2^i = A_2^i$ as it seen from $\mathcal{P}_1$, $\mathcal{P}_2$ terms.

Coefficients at $\mathcal{P}_3$, $\mathcal{P}_4$ give two relations between A_1 and A_2

$$\begin{aligned} A_2^i &= S_3^{-1}(S_1 - \lambda_i) \cdot A_1^i \cdot S_3 (S_2 - \lambda_i)^{-1}, \\ A_2^i &= (S_2 - \lambda_i)^{-1} S_4 \cdot A_1^i \cdot (S_1 - \lambda_i) S_4^{-1}. \end{aligned} \tag{26}$$

Now the matrices A_1, A_2 satisfy both left and right equations

$$\begin{aligned} \hat{O}_i A_1^i &= 0, \qquad & A_1^i \hat{O}'_i &= 0, \\ \hat{O}'_i A_2^i &= 0, \qquad & A_2^i \hat{O}_i &= 0. \end{aligned} \tag{27}$$

Note that homogeneous equations for A_1 lead to following equalities

$$\begin{aligned} S_3^{-1}(S_1 - \lambda_i) \cdot A_1^i &= (S_2 - \lambda_i)^{-1} S_4 \cdot A_1^i, \\ A_1^i \cdot (S_1 - \lambda_i) S_4^{-1} &= A_1^i \cdot S_3 (S_2 - \lambda_i)^{-1}, \end{aligned} \tag{28}$$

so one can see that two relations (26) in fact coincide. Moreover, one can convince yourself that equations for A_2^i (27) are consequence of relation (26) and equations for A_1^i.

Mixing of fermions

Note that the matrix A_1^i has zeroth determinant and may be represented in the form

$$A_1^i = \psi_i(\tilde{\psi}_i)^{\mathrm{T}}, \qquad (29)$$

where vectors ψ_i, $\tilde{\psi}_i$ (columns) are solutions of homogeneous equations

$$\hat{O}_i\psi_i = 0, \quad (\tilde{\psi}_i)^{\mathrm{T}}\hat{O}'_i = 0 \qquad \Big(\text{or} \quad (\hat{O}'_i)^{\mathrm{T}}\tilde{\psi}_i = 0\Big). \qquad (30)$$

Then solution of both left and right eigenstate problems is

$$\Pi_i = \mathcal{P}_1\psi_i(\tilde{\psi}_i)^{\mathrm{T}} + \mathcal{P}_2 S_3^{-1}(S_1-\lambda_i)\psi_i(\tilde{\psi}_i)^{\mathrm{T}}(S_1-\lambda_i)S_4^{-1} - \\ - \mathcal{P}_3\psi_i(\tilde{\psi}_i)^{\mathrm{T}}(S_1-\lambda_i)S_4^{-1} - \mathcal{P}_4 S_3^{-1}(S_1-\lambda_i)\psi_i(\tilde{\psi}_i)^{\mathrm{T}}. \qquad (31)$$

For short notations it is convenient to introduce the vectors ϕ_i, $\tilde{\phi}_i$ as

$$\phi_i = S_3^{-1}(S_1-\lambda_i)\psi_i, \quad (\tilde{\phi}_i)^{\mathrm{T}} = (\tilde{\psi}_i)^{\mathrm{T}}(S_1-\lambda_i)S_4^{-1}. \qquad (32)$$

Recall, that the vectors ϕ_i, $\tilde{\phi}_i$ are solutions of following equations (consequence of definition)

$$\hat{O}'_i\phi_i = 0, \quad (\tilde{\phi}_i)^{\mathrm{T}}\hat{O}_i = 0. \qquad (33)$$

Projectors

In these terms the "matrix"Π_i, which is a solution of both left and right eigenvalue problems, takes very elegant form

$$\Pi_i = \mathcal{P}_1 \cdot \psi_i(\tilde{\psi}_i)^{\mathrm{T}} + \mathcal{P}_2 \cdot \phi_i(\tilde{\phi}_i)^{\mathrm{T}} - \mathcal{P}_3 \cdot \psi_i(\tilde{\phi}_i)^{\mathrm{T}} - \mathcal{P}_4 \cdot \phi_i(\tilde{\psi}_i)^{\mathrm{T}}. \quad (34)$$

Π_i = projectors

Let us require the matrices Π_i (34) to be orthogonal projectors

$$\Pi_i \Pi_k = \delta_{ik} \Pi_k. \tag{35}$$

It leads to the only condition for matrices in (34)

$$\psi_i \left[(\tilde{\psi}_i)^{\mathrm{T}} \psi_k + (\tilde{\phi}_i)^{\mathrm{T}} \phi_k - \delta_{ik} \right] (\tilde{\psi}_k)^{\mathrm{T}} = 0, \tag{36}$$

which is equivalent to the orthonormality condition for vectors involved in (34)

$$(\tilde{\psi}_i)^{\mathrm{T}} \psi_k + (\tilde{\phi}_i)^{\mathrm{T}} \phi_k = \delta_{ik}. \tag{37}$$

- If $i \neq k$ the condition (37) is consequence of equations for ψ_k and $(\tilde{\psi}_i)^{\mathrm{T}}$.
- At $i = k$ (37) defines the normalization (with weight) of the vector ψ_i in respect to $\tilde{\psi}_i$.

CP conservation

In case of CP conservation the self-energy contributions

$$\Sigma(p) = \sum_{M=1}^{4} \mathcal{P}_M \Sigma_M(W) = A(p^2) + \hat{p}B(p^2) + \gamma^5 C(p^2) + \hat{p}\gamma^5 D(p^2) \quad (38)$$

have the following symmetry properties (see, e.g. B. A. Kniehl and A. Sirlin. PR D77 (2008) 116012)

$$A^{\mathrm{T}} = A, \quad B^{\mathrm{T}} = B, \quad D^{\mathrm{T}} = D, \quad C^{\mathrm{T}} = -C, \quad (39)$$

which are equivalent to

$$(\Sigma_{1,2})^{\mathrm{T}} = \Sigma_{1,2}, \quad (\Sigma_3)^{\mathrm{T}} = -\Sigma_4. \quad (40)$$

Since the inverse propagator $S(p)$ has the same symmetry properties (40), it connects matrices $\hat{O}$ and $\hat{O}'$

$$\hat{O}' = -(\hat{O})^{\mathrm{T}}. \quad (41)$$

Eigenprojectors have the form (34) but now two equations (30) coincide

$$\hat{O}_i \psi_i = 0, \quad \hat{O}_i \tilde{\psi}_i = 0. \quad (42)$$

CP conservation

Then (in absence of degeneration) $\tilde{\psi}_i = c\psi_i$ and, redefining vectors $\psi_i \to \sqrt{c}\psi$, we obtain eigenprojectors in the form

$$\Pi_i = \mathcal{P}_1 \cdot \psi_i(\psi_i)^{\mathrm{T}} - \mathcal{P}_2 \cdot \phi_i(\phi_i)^{\mathrm{T}} + \mathcal{P}_3 \cdot \psi_i(\phi_i)^{\mathrm{T}} - \mathcal{P}_4 \cdot \phi_i(\psi_i)^{\mathrm{T}}. \quad (43)$$

Here ψ_i is solution of equation

$$\hat{O}_i\psi_i = 0, \quad (44)$$

vector ϕ_i is related with ψ_i by

$$\phi_i = S_3^{-1}(S_1 - \lambda_i)\psi_i, \quad (\phi_i)^{\mathrm{T}} = -(\psi_i)^{\mathrm{T}}(S_1 - \lambda_i)S_4^{-1} \quad (45)$$

and satisfies the homogenious equation $\hat{O}'_i\phi_i = 0$.
The orthonormality condition $\Pi_i\Pi_k = \delta_{ik}\Pi_k$ leads to

$$(\psi_i)^{\mathrm{T}}\psi_k - (\phi_i)^{\mathrm{T}}\phi_k = \delta_{ik}. \quad (46)$$

As it was said before, it follows from homogeneous equation.

Multiplicative renormalization of matrix propagator

It is convenient to renormalize inverse matrix propagator $S(p)$

$$S^{\mathrm{r}}(p) = \bar{Z} S(p) Z = \sum_{M=1}^{4} \bar{Z} S_M(W) \mathcal{P}_M Z. \tag{47}$$

To obtain correct properties of matrix components $S^{\mathrm{r}}_M(W)$ at pole vicinity, one needs two matrix renormalization "constants"

$$Z = \alpha + \beta\gamma^5, \quad \bar{Z} = \bar{\alpha} + \bar{\beta}\gamma^5, \tag{48}$$

where α, β, $\bar{\alpha}$ and $\bar{\beta}$ are some $n \times n$ matrices. The choice $\bar{\alpha} = \alpha^{\mathrm{T}}$ and $\bar{\beta} = -\beta^{\mathrm{T}}$ allows to preserve CP-invariance of renormalized matrix propagator.

Requirements for renormalization

The renormalized dressed matrix propagator $G^{\mathrm{r}}(p)$ has poles at points $\{m_k\}$, which are zeroes of eigenvalues: $\lambda_k(W = m_k) = 0$. In vicinity of point $W = m_k$ matrix propagator has the form (K.I.Aoki et al. Prog. Theor. Phys. Suppl. 73 (1982) 1.)

$$G^{\mathrm{r}}(p) \sim \begin{pmatrix} & \vdots & \\ \cdots & \dfrac{1}{\hat{p} - m_k} & \cdots \\ & \vdots & \end{pmatrix}, \tag{49}$$

where other elements of $G^{\mathrm{r}}(p)$ are regular at $W = m_k$.

Requirements for renormalization

Inverse matrix propagator $S^{\mathrm{r}}(p) = \sum_{M=1}^{4} S^{\mathrm{r}}_M(W)\mathcal{P}_M$ looks like

$$S^{\mathrm{r}}_1 \sim \begin{pmatrix} O(1) & \ldots & O(\epsilon_k) & \ldots & O(1) \\ \vdots & & \vdots & & \vdots \\ O(\epsilon_k) & \ldots & \epsilon_k & \ldots & O(\epsilon_k) \\ \vdots & & \vdots & & \vdots \\ O(1) & \ldots & O(\epsilon_k) & \ldots & O(1), \end{pmatrix}, \quad \text{at } \epsilon_k = W - m_k \to 0, \quad S^{\mathrm{r}}_2 \sim O(1),$$

$$S^{\mathrm{r}}_3 \sim \begin{pmatrix} & & O(1) & & \\ & & \vdots & & \\ O(\epsilon_k) & \ldots & 0 & \ldots & O(\epsilon_k) \\ & & \vdots & & \\ & & O(1) & & \end{pmatrix}, \quad S^{\mathrm{r}}_4 \sim \begin{pmatrix} & & O(\epsilon_k) & & \\ & & \vdots & & \\ O(1) & \ldots & 0 & \ldots & O(1) \\ & & \vdots & & \\ & & O(\epsilon_k) & & \end{pmatrix}. \tag{50}$$

Solution for renormalization constants

And answer for renormalization constants in two rows.

One can check that matrices α, β have to be chosen in form

$$\begin{aligned} \alpha &= \Big(\psi_1(m_1), \psi_2(m_2), \dots, \psi_n(m_n)\Big), \\ \beta &= -\Big(\phi_1(m_1), \phi_2(m_2), \dots, \phi_n(m_n)\Big), \end{aligned} \tag{51}$$

where $\psi_k(m_k)$ $(\phi_k(m_k))$ denotes column of components of vector $\psi_k(m_k)$ $(\phi_k(m_k))$.

Conclusions

- We developed non-standard approach to problem of fermion mixing. In fact it is based on QFT but uses another algebraical construction of martix propagator — generalization of well-known spectral representation of operator. Note that this representation becomes non-trivial in presence of P-parity violation.
- This approach separates different poles in a matrix propagator (in particular, poles with positive and negative energy) and looks very natural for mixing problem. For example, the renormalization constants are expressed just in introduced by us terms and looks very simple.
- There exist few interesting question related with mixing problem in QFT. One of them — properties of spin projectors for dressed fermions, which are non-trivial in case of P-parity violation.

Thank you for attention