Generalized Beth-Uhlenbeck approach for hadrons in quark matter

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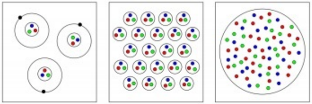
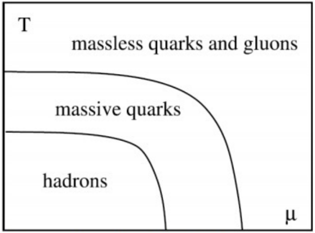
September 17, 2014

The XXII International Baldin Seminar on High Energy Physics Problems "Relativistic Nuclear Physics and QCD"

D. Blaschke, M. Buballa, A. Dubinin, G. Röpke, D. Zablocki, Annals Phys. 348 (2014) 228; [arXiv:1305.3907 [hep-ph]].

Introduction



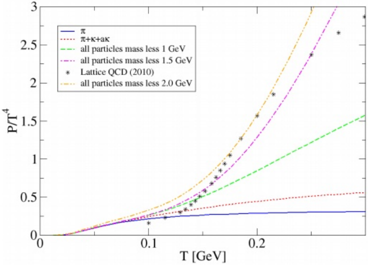


H. Satz, Lect. Notes Phys. **841** (2012) 1.

The partition functional for an ideal gas of particles of mass m at temperature T

$$\ln \mathcal{Z}[T, V, \mu] = -sg \frac{V}{2\pi^2} \int_0^\infty dpp^2 \times$$
$$\times \ln \left[1 - se^{-(\sqrt{p^2+m^2}-mu)/T}\right]$$

Introduction



The partition functional for an ideal gas of particles of mass m at temperature T

$$\ln \mathcal{Z}_i[T, V, \mu] = -sg\frac{V}{2\pi^2}\int_0^\infty dpp^2 \times$$

$$\times \ln\left[1 - se^{-(\sqrt{p^2+m^2}-mu)/T}\right]$$

$$\Omega_i[T, \mu] = \frac{T}{V}\ln \mathcal{Z}[T, V, \mu]$$

$$P[T, \mu] = -\sum \Omega_i[T, \mu]$$

NJL model

NJL model Lagrangian , with $N_f = 2$ and $N_c = 3$

$$\mathscr{L} = \bar{q}(\mathrm{i}\partial\!\!\!/ - m_0 + \mu\gamma_0)q + \mathscr{L}_{\mathrm{int}} \ ,$$

$$\mathscr{L}_{\mathrm{int}} = G_{\mathrm{S}}\left[(\bar{q}q)^2 + (\bar{q}\mathrm{i}\gamma_5\vec{\tau}q)^2\right] + G_{\mathrm{D}} \sum_{A=2,5,7} (\bar{q}\mathrm{i}\gamma_5\tau_2\lambda_A q^c)(\bar{q}^c\mathrm{i}\gamma_5\tau_2\lambda_A q) \ .$$

Thermodynamic potential and grand partition function

$$\Omega(T,\mu) = -\frac{T}{V}\ln\mathcal{Z}(T,\mu) \ , \mathcal{Z} = \int \mathscr{D}q\mathscr{D}\bar{q}\exp\left[\int \mathrm{d}^4x_E \ \mathscr{L}\right] \ ,$$

$$\int \mathrm{d}^4x_E = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x}$$

Bosonization

Nambu-Gorkov bispinors

$$\Psi \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} q \\ q^c \end{pmatrix} \qquad \bar{\Psi} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{q} & \bar{q}^c \end{pmatrix}$$

$$\mathcal{Z} = \int \mathscr{D}\sigma \mathscr{D}\vec{\pi} \mathscr{D}\Delta_A \mathscr{D}\Delta_A^* \exp\left[\int \mathrm{d}^4 x_E \left(-\frac{\sigma^2 + \vec{\pi}^2}{4G_S} - \frac{\Delta_A^* \Delta_A}{4G_D}\right)\right] \times$$
$$\int \mathscr{D}\Psi \mathscr{D}\bar{\Psi} \exp\left[\int \mathrm{d}^4 \mathbf{x_E}\, \bar{\Psi}\, S^{-1} \Psi\right] ,$$

$$S^{-1} \equiv \begin{pmatrix} \mathrm{i}\partial\!\!\!/ + \mu\gamma_0 - m_0 - \sigma - \mathrm{i}\gamma_5 \vec{\tau}\cdot\vec{\pi} & \mathrm{i}\Delta_A \gamma_5 \tau_2 \lambda_A \\ \mathrm{i}\Delta_A^* \gamma_5 \tau_2 \lambda_A & \mathrm{i}\partial\!\!\!/ - \mu\gamma_0 - m_0 - \sigma - \mathrm{i}\gamma_5 \vec{\tau}^T \cdot \vec{\pi} \end{pmatrix} .$$

Bosonization

Integrated out the bispinor fields

$$\mathscr{Z} \;=\; \int \mathscr{D}\sigma \mathcal{D}\vec{\pi} \mathscr{D}\Delta_A \mathcal{D}\Delta_A^* \, \mathrm{e}^{-\int \mathrm{d}^4 x_E \, \left\{ \frac{\sigma^2+\vec{\pi}^2}{4G_\mathrm{S}} + \frac{|\Delta_A|^2}{4G_\mathrm{D}} \right\} + \frac{1}{2} \ln \det\left(\beta S^{-1}\right)} \; .$$

After Fourier transforming the inverse quark propagator has form

$$S^{-1} = \begin{pmatrix} (\mathrm{i}z_n + \mu^*)\gamma_0 - m - \mathrm{i}\gamma_5 \vec{\tau} \cdot \vec{\pi} & \Delta_A \mathrm{i}\gamma_5 \tau_2 \lambda_A \\ \Delta_A^* \mathrm{i}\gamma_5 \tau_2 \lambda_A & (\mathrm{i}z_n - \mu^*)\gamma_0 - m - \mathrm{i}\gamma_5 \vec{\tau}^{\,T} \cdot \vec{\pi} \end{pmatrix}$$

The combinations $m = m_0 + \sigma$ which can be interpreted as an effective (constituent) quark mass. Mean-field $\sigma = \sigma_\mathrm{MF}$.

Beyond mean field: gaussian approximation

Shifted fields is introduced,

$$\sigma \quad \to \quad \sigma_{\rm MF} + \sigma \ ,$$

In addition, the fluctuations contribute to the partition function via the inverse propagator, which can be written as

$$S^{-1} = S_{\rm MF}^{-1} + \Sigma \ ,$$

with

$$\Sigma \quad = \quad \begin{pmatrix} -\sigma - \mathrm{i}\gamma_5 \vec{\tau}\cdot\vec{\pi} & \delta_A \mathrm{i}\gamma_5 \tau_2 \lambda_A \\ \delta_A^* \mathrm{i}\gamma_5 \tau_2 \lambda_A & -\sigma - \mathrm{i}\gamma_5 \vec{\tau}^{\,T}\cdot\vec{\pi} \end{pmatrix} \ .$$

The logarithmic term in $\mathscr{Z}$ then takes the form

$$\begin{aligned} \ln\det\left(\beta S^{-1}\right) &= \mathrm{Tr}\ln\left(\beta S_{\rm MF}^{-1} + \beta\Sigma\right) = \\ &= \mathrm{Tr}\ln\left(\beta S_{\rm MF}^{-1}\right) + \underbrace{\mathrm{Tr}\ln(1 + S_{\rm MF}\Sigma)}_{\mathrm{Tr}\left(S_{\rm MF}\Sigma - \frac{1}{2}S_{\rm MF}\Sigma S_{\rm MF}\Sigma\right) + \mathscr{O}(\Sigma^3)} \end{aligned}$$

gaussian approximation

partition function in gaussian approximation

$$\mathscr{Z}_{\mathrm{Gauß}} = \mathscr{Z}_{\mathrm{MF}} \int \mathscr{D}\sigma \mathscr{D}\vec{\pi} \mathscr{D}\delta_A \mathscr{D}\delta_A^* \, \mathrm{e}^{\mathscr{A}^{(1)}+\mathscr{A}^{(2)}} \; ,$$

with the mean-field partition function $\mathscr{Z}_{\mathrm{MF}} = \exp\left(-\beta V \Omega_{\mathrm{MF}}\right)$

$$\mathscr{A}^{(1)} = -\beta V \left(\frac{\sigma_{\mathrm{MF}}\sigma}{2G_{\mathrm{S}}} + \frac{\Delta_{\mathrm{MF}}\delta_2^* + \Delta_{\mathrm{MF}}^*\delta_2}{4G_{\mathrm{D}}} \right) + \frac{1}{2} \operatorname{Tr}\left[S_{\mathrm{MF}}\Sigma\right] \; ,$$

$$\mathscr{A}^{(2)} = -\beta V \left(\frac{\sigma^2 + \vec{\pi}^2}{4G_{\mathrm{S}}} + \frac{|\delta_A|^2}{4G_{\mathrm{D}}} \right) - \frac{1}{4} \operatorname{Tr}\left[S_{\mathrm{MF}}\Sigma S_{\mathrm{MF}}\Sigma\right] \; .$$

gaussian approximation

$$\mathscr{Z}_{\text{Gauß}} = \mathscr{Z}_{\text{MF}} \int \mathcal{D}\sigma \mathcal{D}\vec{\pi} \mathcal{D}\delta_A \mathcal{D}\delta_A^* \, \mathrm{e}^{\mathscr{A}^{(2)}} \ .$$

We combine all fields in a vector

$$X = \begin{pmatrix} \vec{\pi} \\ \sigma \\ \delta_A \\ \delta_A^* \end{pmatrix} \ , \quad X^\dagger = (\vec{\pi}^{\,T}, \sigma, \delta_A^*, \delta_A) \ ,$$

The trace in the exponent can then be written as

$$\frac{1}{2} \operatorname{Tr}[S_{\text{MF}} \Sigma S_{\text{MF}} \Sigma] = -X^\dagger \Pi X \ ,$$

Combining both parts

$$\frac{\sigma^2 + \vec{\pi}^2}{2G_{\text{S}}} + \frac{|\delta_A|^2}{2G_{\text{D}}} + \frac{1}{2} \frac{T}{V} \operatorname{Tr}[S_{\text{MF}} \Sigma S_{\text{MF}} \Sigma] = X^\dagger \tilde{S}^{-1} X \ ,$$

gaussian approximation

$$\begin{aligned}\mathscr{Z}_{\mathrm{Gauß}} &= \mathscr{Z}_{\mathrm{MF}} \int \mathscr{D}X \, \mathrm{e}^{-\frac{1}{2}\int \mathrm{d}^4 x_E \, \left\{X^\dagger \tilde{S}^{-1} X\right\}} \\ &= \underbrace{\mathscr{Z}_{\mathrm{MF}} \left[\det\left(\beta^2 \tilde{S}^{-1}\right)\right]^{-1/2}}_{\text{After diagonalizing } \tilde{\mathrm{S}}^{-1}} = \mathscr{Z}_{\mathrm{MF}} \prod_{\mathrm{X}} \mathscr{Z}_{\mathrm{X}} \,,\end{aligned}$$

$$\Omega_{\mathrm{Gauß}} = -\frac{T}{V} \ln \mathscr{Z}_{\mathrm{Gauß}} = \underbrace{\Omega_{\mathrm{MF}}}_{\Omega_{\mathrm{cond}} + \Omega_{\mathrm{Q}}} + \sum_{\mathrm{X=M,\ D,\ \bar{D},\ ...}} \Omega_{\mathrm{X}} \,,$$

$$\Omega_{\mathrm{X}}(T,\mu) = \frac{d_X}{2} \frac{T}{V} \operatorname{Tr} \ln\left[\beta^2 S_{\mathrm{X}}^{-1}(\mathrm{i}z_n, \mathbf{q})\right] = \frac{d_{\mathrm{X}}}{2} T \sum_n \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \ln\left[\beta^2 S_{\mathrm{X}}^{-1}(\mathrm{i}z_n, \mathbf{q})\right] .$$

Generalized Beth-Uhlenbeck equation of state

$$S_{\rm X}^{-1}({\rm i}z_n,\mathbf{q}) = G_{\rm X}^{-1} - \Pi_{\rm X}({\rm i}z_n,\mathbf{q}),\ \ S_{\rm X} = |S_{\rm X}|\,{\rm e}^{{\rm i}\delta_{\rm X}} = S_R + {\rm i}S_I\ ,$$

$$\delta_{\rm X}(\omega,\mathbf{q}) = -\,{\rm Im}\,\ln\left[\beta^2 S_{\rm X}^{-1}(\omega-\mu_X+{\rm i}\eta,\mathbf{q})\right]\ ,$$

$$\begin{aligned}
p_{\rm X} &= -\Omega_{\rm X}(T,\mu) = -\frac{d_{\rm X}}{2}T\sum_n\int\frac{{\rm d}^3q}{(2\pi)^3}\ln\left[\beta^2 S_{\rm X}^{-1}({\rm i}z_n,\mathbf{q})\right]\ ,\\
&= \frac{d_{\rm X}}{2}T\sum_n\int\frac{{\rm d}^3q}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{{\rm d}\omega}{2\pi}\,\frac{2}{{\rm i}z_n-\omega}\,{\rm Im}\ln\left[\beta^2 S_{\rm X}^{-1}(\omega+{\rm i}\eta,\mathbf{q})\right]\ ,\\
&= -\frac{d_{\rm X}}{2}T\sum_n\int\frac{{\rm d}^3q}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{{\rm d}\omega}{2\pi}\,\frac{2}{{\rm i}z_n-(\omega-\mu_X)}\delta_X(\omega,\mathbf{q})\ ,\\
&= d_{\rm X}\int\frac{{\rm d}^3q}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{{\rm d}\omega}{2\pi}\,n_X^-(\omega)\delta_X(\omega,\mathbf{q})\ ,\ \left(\ n_X^-(-\omega) = -[1+n_X^+(\omega)]\right)\\
&= d_{\rm X}\int\frac{{\rm d}^3q}{(2\pi)^3}\int_0^{\infty}\frac{{\rm d}\omega}{2\pi}\,[1+n_X^-(\omega)+n_X^+(\omega)]\delta_{\rm X}(\omega,\mathbf{q})\ .
\end{aligned}$$

$$n_X^{\pm}(\omega) = 1/\{\exp[(\omega\pm\mu_X)/T]-1\}$$

Phase shift

$$\Pi_{\mathrm{X}}(z,\mathbf{q}) = \Pi_{X,0} + \Pi_{X,2}(z,\mathbf{q}) \ ,$$

$$S_{\mathrm{X}}(z-\mu_X+\mathrm{i}\eta,\mathbf{q}) = \frac{1}{G_{\mathrm{X}}^{-1} - \Pi_{X,0} - \Pi_{X,2}(z-\mu_X+\mathrm{i}\eta,\mathbf{q})} = \frac{1}{\Pi_{X,2}(z-\mu_X+\mathrm{i}\eta,\mathbf{q})}\,\frac{1}{R_X(z^2,\mathbf{q})-1} \ ,$$

where the auxiliary function

$$R_X(z^2,\mathbf{q}) = \frac{1-G_{\mathrm{X}}\Pi_{X,0}}{G_{\mathrm{X}}\Pi_{X,2}(z-\mu_X+\mathrm{i}\eta,\mathbf{q})}$$

$$\ln S_{\mathrm{X}}(z-\mu_X+\mathrm{i}\eta,\mathbf{q})^{-1} = \ln \Pi_{X,2}(z-\mu_X+\mathrm{i}\eta,\mathbf{q}) + \ln[R_X(z^2,\mathbf{q})-1] \ ,$$

$$\delta_{\mathrm{X}}(\omega,\mathbf{q}) = \delta_{X,c}(\omega,\mathbf{q}) + \delta_{X,R}(\omega,\mathbf{q}) \ ,$$

where

$$\begin{aligned}
\delta_{X,c}(\omega,\mathbf{q}) &= -\arctan\left(\frac{\mathrm{Im}\,\Pi_{X,2}(\omega-\mu_X+\mathrm{i}\eta,\mathbf{q})}{\mathrm{Re}\,\Pi_{X,2}(\omega-\mu_X+\mathrm{i}\eta,\mathbf{q})}\right) \ ,\\
\delta_{X,R}(\omega,\mathbf{q}) &= \arctan\left(\frac{\mathrm{Im}\,R_X(\omega^2,\mathbf{q})}{1-\mathrm{Re}\,R_X(\omega^2,\mathbf{q})}\right) \ .
\end{aligned}$$

Phase shift

$$
\begin{aligned}
1 - \text{Re}\, R_X(\omega^2) &\approx \underbrace{1 - \text{Re}\, R_X(\omega_X^2)}_{=0} - (\omega^2 - \omega_X^2)\, \text{Re}\, \frac{dR_X(z^2)}{dz^2}\bigg|_{z=\omega_X} , \qquad (1) \\
\text{Im}\, R_X(\omega^2) &\approx \text{Im}\, R_X(\omega_X^2) \; .
\end{aligned}
$$

From this follows

$$
\frac{1 - \text{Re}\, R_X(\omega^2)}{\text{Im}\, R_X(\omega^2)} \approx -(\omega^2 - \omega_X^2) \frac{\text{Re}\, \frac{dR_X(z^2)}{dz^2}\Big|_{z=\omega_X}}{\text{Im}\, R_X(\omega_X^2)} \; .
$$

If we now define that

$$
\omega_X \Gamma_X = - \frac{\text{Im}\, R_X(\omega_X^2)}{\text{Re}\, \frac{dR_X(z^2)}{dz^2}\Big|_{z=\omega_X}} \; ,
$$

the resonant phase shift becomes

$$
\delta_{X,R}(\omega, \mathbf{q}) = \arctan\left(\frac{\omega_X \Gamma_X}{\omega^2 - \omega_X^2} \right) \; ,
$$

which corresponds to the Breit-Wigner form for the spectral density in the Beth-Uhlenbeck EoS

$$
\frac{d\delta_{X,R}(\omega)}{d\omega} = \frac{2\omega\omega_{\mathrm{X}}\Gamma_{\mathrm{X}}}{(\omega^2 - \omega_X^2)^2 + \omega_{\mathrm{X}}^2 \Gamma_{\mathrm{X}}^2} \; .
$$

Phase shift

This form goes over to the spectral density of a bound state when the width parameter $\Gamma_{\rm X} \to 0$,

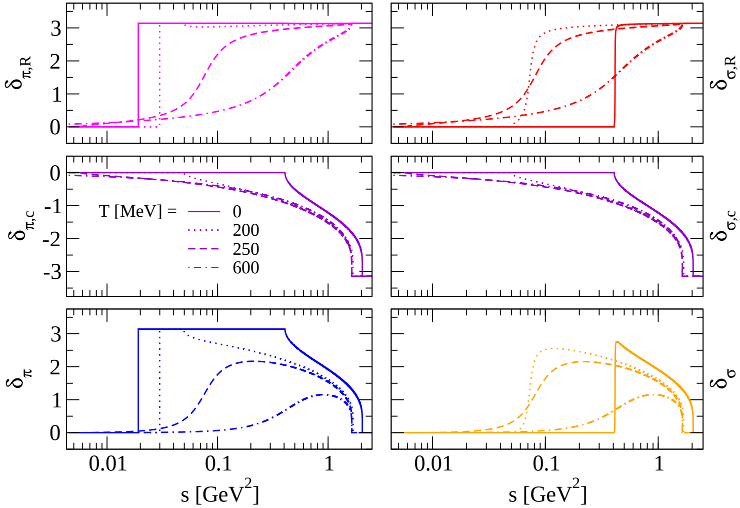
$$\lim_{\Gamma_{\rm X}\to 0} \frac{d\delta_{X,R}(\omega)}{d\omega} = \pi \left[\delta(\omega - \omega_X) + \delta(\omega + \omega_X)\right] \ ,$$

Levionson theorem

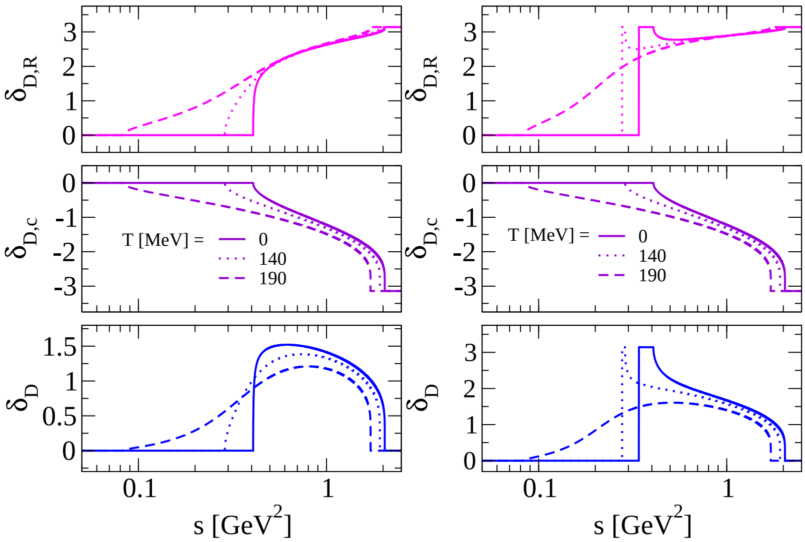
$$\pi n_{B,X} = -[\delta_X(\infty) - \delta_X(\omega_{\mathrm{thr}})]$$

$$\begin{aligned}\int_0^\infty d\omega \frac{1}{\pi}\frac{d\delta_X(\omega;T)}{d\omega} &= 0 = \underbrace{\int_0^{\omega_{\mathrm{thr}}(T)} d\omega \frac{1}{\pi}\frac{d\delta_X(\omega;T)}{d\omega}}_{n_{B,X}(T)} \\ &+ \underbrace{\frac{1}{\pi}\int_{\omega_{\mathrm{thr}}(T)}^\infty d\omega \frac{d\delta_X(\omega;T)}{d\omega}}_{\frac{1}{\pi}[\delta_X(\infty;T)-\delta_X(\omega_{\mathrm{thr}};T)]} \ .\end{aligned}$$

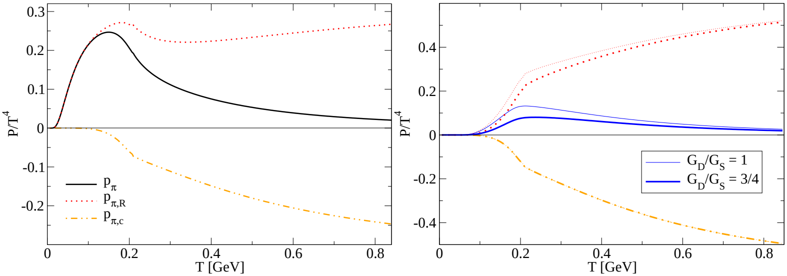
Results: phase shift of pion and σ-meson



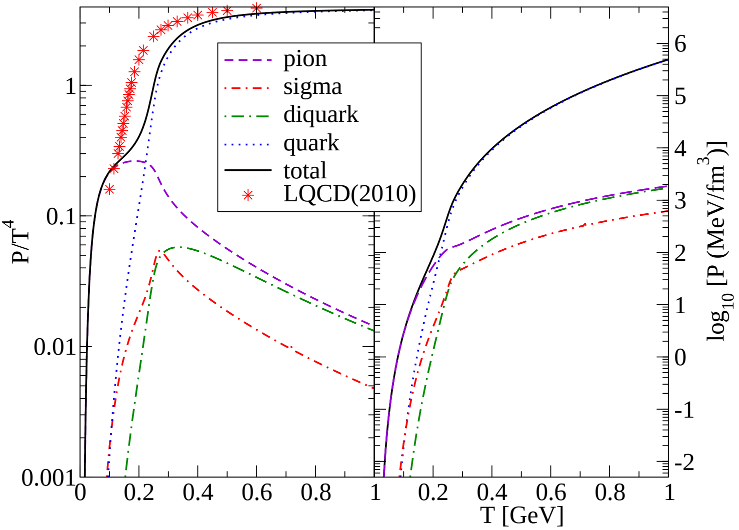
Results: phase shift of diquark



Results: Pressure



Results: Pressure



Conclusion

- Mott mechanism for hadron dissociation is demonstrated
- Beth-Uhlenbeck formalism is developed for EoS, describing hadronic correlations by phase shifts
- Phase shifts are evaluated with T-dependence for pion, σ-meson, diquark and validity of the Levinson theorem is demonstrated
- outlook: include more hadronic states into this description in order to obtain accordance with Lattice QCD thermodynamics