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Solving Bethe-Salpeter equation for scattering states

V.A. Karmanov[†] and J. Carbonell[‡]

[†] *Lebedev Physical Institute, Moscow, Russia*

[‡] *CEA, Saclay, France*

• BS equation for the scattering states

E.E. Salpeter and H. Bethe, Phys. Rev. **84**, 1232 (1951)

$$
\begin{aligned}
F(p,p'',P) \quad &= \quad V^{inh}(p,p'',P) - i\int \frac{d^4p'}{(2\pi)^4} \\
&\times \quad \frac{V(p,p',P)F(p',p'',P)}{\left[\left(\frac{1}{2}P+p'\right)^2 - m^2 + i\epsilon\right]\left[\left(\frac{1}{2}P-p'\right)^2 - m^2 + i\epsilon\right]}
\end{aligned}
$$

Ladder kernel: $\quad V(p,p',P) = -\dfrac{g^2}{(p-p')^2-\mu^2+i\epsilon}$

p – relative 4-momentum (variable).

p' – integration 4-momentum (variable).

p'' – physical (relative) 4-momentum.

P – total 4-momentum.

It determines the off-shell amplitude in Minkowski space.

In c.m.-frame $\vec{P}=0$: $F=F(p_0,p;p'')$ depends on

two variables p_0, p (for S-wave).

On-mass shell: $F^{on}=F(p_0=0,p=p'';p'')=F(p'')$

– physical amplitude.

• Separable kernel

BS equation was solved, in Minkowski space,
for a separable kernel

V. Burov, S. Bondarenko, E. Rogochaya.
Tjon et al.

Off-mass shell amplitude is found
(and the on-mass shell one – the phase shifts)

• Ladder kernel – Field theory based kernel

BS equation was solved for the ladder kernel,

by Wick rotation $p_0 = ip_4$, i.e., in Euclidean space.

Schwartz and Zemach (1966); Levine, Tjon, Wright (1966); Haymaker (1967); Maris et al. (2002)

In this way, one obtains the Euclidean amplitude $F_E(p_4, p; p'')$.

On-mass shall: $p_0 = ip_4 = 0, \quad p = p''$.

Hence: $F^{on} = F_M(p_0 = 0, p = p''; p'') = F_E(p_4 = 0, p = p''; p'')$.

Therefore, the Euclidean solution indeed gives

on-shell amplitude – the physical phase shifts.

However, the BS equation has yet never been solved for ladder kernel, for the off-shell amplitude $F(p_0, p; p'')$ itself which enters this equation.

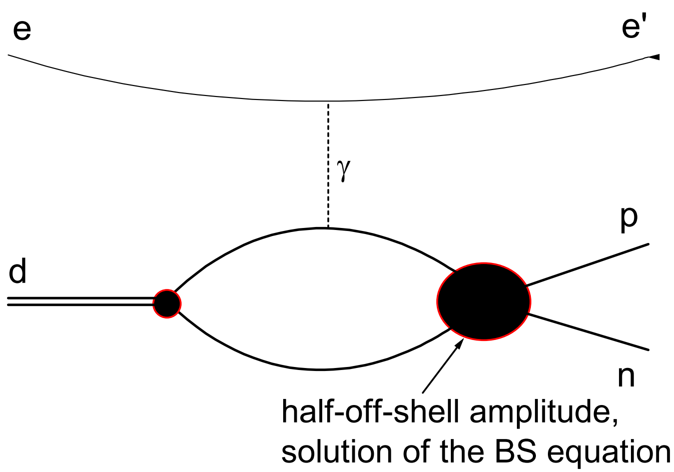
Why? – Because of singularities.

We need the

off-shell BS amplitude in Minkowski space

to calculate the transition form factor $ed \to enp$,

or as an input for the three-body BS-Faddeev equations.



• Outline

- Solving the scattering BS equation in Minkowski space.
- Phase shifts.
- Scattering length.
- Inelasticity (above threshold).
- Off-mass shell amplitude
  (main aim of the present work).

• Methods in Minkowski space

1. Nakanishi integral representation.

- Was applied to solution of the bound state problem in V.A. Karmanov and J. Carbonell, Eur. Phys. J. **A27** (2006) 1.
- For the scattering states, a formalism is developed in T. Frederico, G. Salmè, and M. Viviani, Phys. Rev. D 85 (2012) 036009. The numerical solution was not yet obtained.

2. Direct and accurate treating of singularities.

–Method we develop in this work.

• Four sources of singularities

$$F(p,p'',P) = V^{inh}(p,p'',P) - i\int \frac{d^4p'}{(2\pi)^4}$$
$$\times \frac{V(p,p',P)F(p',p'',P)}{\left[\left(\frac{1}{2}P+p'\right)^2 - m^2 + i\epsilon\right]\left[\left(\frac{1}{2}P-p'\right)^2 - m^2 + i\epsilon\right]}$$

1. Constituent propagators $\frac{1}{\left[\left(\frac{1}{2}P+p'\right)^2-m^2+i\epsilon\right]\left[\left(\frac{1}{2}P-p'\right)^2-m^2+i\epsilon\right]}$
2. Kernel $V(p,p',P) = -\frac{g^2}{(p-p')^2-\mu^2+i\epsilon}$
3. Inhomogeneous term $V^{inh}(p,p'',P) = -\frac{g^2}{(p-p'')^2-\mu^2+i\epsilon}$
4. Amplitude $F(p',p'',P)$ itself

We treat all the singularities in a way which allows to calculate all the singular integrals numerically.

How?

• 1. Constituent propagators

In c.m.-frame $\vec{P}=0$, $P_0 \equiv \sqrt{s} = 2\varepsilon_{p''}$:

$$F(p_0,p,u;p'') = V^{inh}(p_0,p,u;p'') - i \int \frac{d^3p'}{(2\pi)^4} \int_{-\infty}^{\infty} dp'_0$$

$$\times \quad \frac{V(p_0,p,u;p'_0,p',u')F(p'{}_0,p',u';p'')}{(p'_0-p_0^- + i\epsilon)(p'_0+p_0^- - i\epsilon)(p'_0+p_0^+ - i\epsilon)(p'_0-p_0^+ + i\epsilon)}$$

with $p_0^+ = \varepsilon_{p'} + \varepsilon_{p''}, \quad p_0^- = \varepsilon_{p'} - \varepsilon_{p''}.$

Product of four pole terms

$$\frac{1}{(p_0' - p_0^- - i\epsilon)} = PV \frac{1}{(p_0' - p_0^-)} + i\pi\delta(p_0' - p_0^-)$$

etc. (for all four products).

$$\int dp' dp_0' \, (PV + i\delta) \, (PV + i\delta) \, (PV + i\delta) \, (PV + i\delta)$$

$$= \int dp' dp_0' PV \cdot PV \cdot PV \cdot PV \quad \Leftarrow \textbf{2D integral}$$

$$+ \int dp' dp_0' PV \cdot PV \cdot PV \cdot \delta + \ldots \quad \Leftarrow \textbf{1D integral}$$

$$+ \int dp' dp_0' PV \cdot PV \cdot \delta \cdot \delta \quad \Leftarrow \textbf{0D integral}$$

• BS equation for the S-wave amplitude

$$F_0(p_0,p;p'') = V_0^{inh}(p_0,p;p'')$$

$$+\int_0^\infty \frac{dp'}{\varepsilon_{p'}}\left\{\frac{i}{4\varepsilon_{p''}}\int_0^\infty \frac{dp'_0}{(p'^2_0 - p_0^{-\,2})}\left[V_0^s(p_0,p;p'_0,p')F_0(p'_0,p';p'')\right.\right.$$

$$\left. -V_0^s(p_0,p;p_0^-,p')F_0(|p_0^-|,p';p'')\right]$$

$$-\frac{i}{4\varepsilon_{p''}}\int_0^\infty \frac{dp'_0}{(p'^2_0 - p_0^{+\,2})}\left[V_0^s(p_0,p;p'_0,p')F_0(p'_0,p';p'')\right.$$

$$\left.\left. -V_0^s(p_0,p;p_0^+,p')F_0(p_0^+,p';p'')\right]\right\} \quad \Leftarrow \textbf{2D integral}$$

$$+\int_0^\infty \frac{dp'}{\varepsilon_{p'}}\left\{\frac{\pi}{8\varepsilon_{p''}}\frac{1}{(\varepsilon_{p'}-\varepsilon_{p''})}\left[V_0^s(p_0,p;p_0^-,p')F_0(|p_0^-|,p';p'')\right.\right.$$

$$- \quad \left.\frac{2\varepsilon_{p'}}{(\varepsilon_{p'}+\varepsilon_{p''})}V_0^s(p_0,p;p'_0=0,p'=p'')F_0(p'_0=0,p'=p'';p'')\right]$$

$$\left.-\frac{\pi}{8\varepsilon_{p''}}\frac{1}{(\varepsilon_{p'}+\varepsilon_{p''})}V_0^s(p_0,p;p_0^+,p')F_0(p_0^+,p';p'')\right\} \quad \Leftarrow \textbf{1D integral}$$

$$+\frac{i\pi^2}{8p''\varepsilon_{p''}}V_0^s(p_0,p;p'_0=0,p'=p'')F_l(p'_0=0,p'=p'';p'') \quad \Leftarrow \textbf{0D}$$

• 2. S-wave kernel

$$
\begin{aligned}
V_0(p_0,p,p_0',p') &= -\int_{-1}^{1} \frac{g^2\,du}{(p_0-p_0')^2-(p^2-2p\,p'\,u+p'^2)-\mu^2+i\epsilon} \\
&= -\frac{8\pi m^2\alpha}{pp'}\int_{-1}^{1}\frac{du}{\eta+u+i\epsilon} \\
&= -\frac{8\pi\alpha m^2}{pp'}\log\frac{|\eta+1|}{|\eta-1|}+\frac{i8\pi\alpha m^2}{pp'}U(\eta)
\end{aligned}
$$

where

$$
\alpha=\frac{g^2}{16\pi m^2},\qquad \eta=\frac{(p_0-p_0')^2-p^2-p'^2-\mu^2}{2pp'}
$$

and

$$
U(\eta)=\begin{cases} 1, & \text{if } |\eta|\le 1 \\ 0, & \text{if } |\eta|>1 \end{cases}
$$

Singularities of kernel

Kernel is singular when $\eta = \pm 1$. That is:

$$(p_0 - p'_0)^2 - (p \mp p')^2 - \mu^2 = 0$$

–Moving singularities in $p'_0 = p'_0(p')$.

In addition: 4 quadratic equations $\rightarrow$ 8 singularities.

$$(p_0 - p_0^{\pm})^2 - (p \pm p')^2 - \mu^2 = 0, \quad p_0^{\pm} = \varepsilon_{p'} \pm \varepsilon_{p''}$$

–Fixed singularities in p'.

All of them are $\log$-singularities.

To improve precision, we integrate numerically from one singularity to other.

$$\int_0^\infty \dots dp' = \int_0^{p_1^{sing}} \dots dp' + \int_{p_1^{sing}}^{p_2^{sing}} \dots dp' + \int_{p_2^{sing}}^{p_3^{sing}} \dots dp' + \dots$$

In this integration we use appropriate change of variables.

• 3. Inhomogeneous term V^{inh}

The pole singularity which becomes the log-singularity for the partial wave.

• 4. Amplitude F itself

$$F = V^{inh} + V\Pi V^{inh} + V\Pi V^{inh}\Pi V^{inh} + \ldots$$

Amplitude F contains the singularities of each its iterative term.

The most dangerous ones result from the inhomogeneous term.

Introduce new function f:

$$F = \gamma V^{inh} f$$

γ is an arbitrary smooth function.

Inhomogeneous term in the equation for f is smooth.

$\Rightarrow$ f is now also smooth.

We solve equation for f.

The equation is lengthy,

but now the integrand is smooth, the integrals are easy computed!

We find numerical solution, decomposing it in the spline basis.

• Results for bound states

We reproduce the binding energies found previously, by other methods.

$$
\begin{aligned}
& B(\text{Minkowski space, present solution}) \\
= \; & B(\text{Nakanishi representation}) \\
= \; & B(\text{Euclidean space})
\end{aligned}
$$

The method works for the bound states!

• Extracting phase shift

$$F^{on} = F_l(p_0 = 0, p = p''; p'') \quad \Leftarrow$$ on-mass shell

$$S_l = e^{i2\delta_l} = 1 + \frac{2ip''F^{on}}{\varepsilon_{p''}}$$

Or:

$$\delta_l = \frac{1}{2i} \log\left(1 + \frac{2ip''F^{on}}{\varepsilon_{p''}}\right)$$

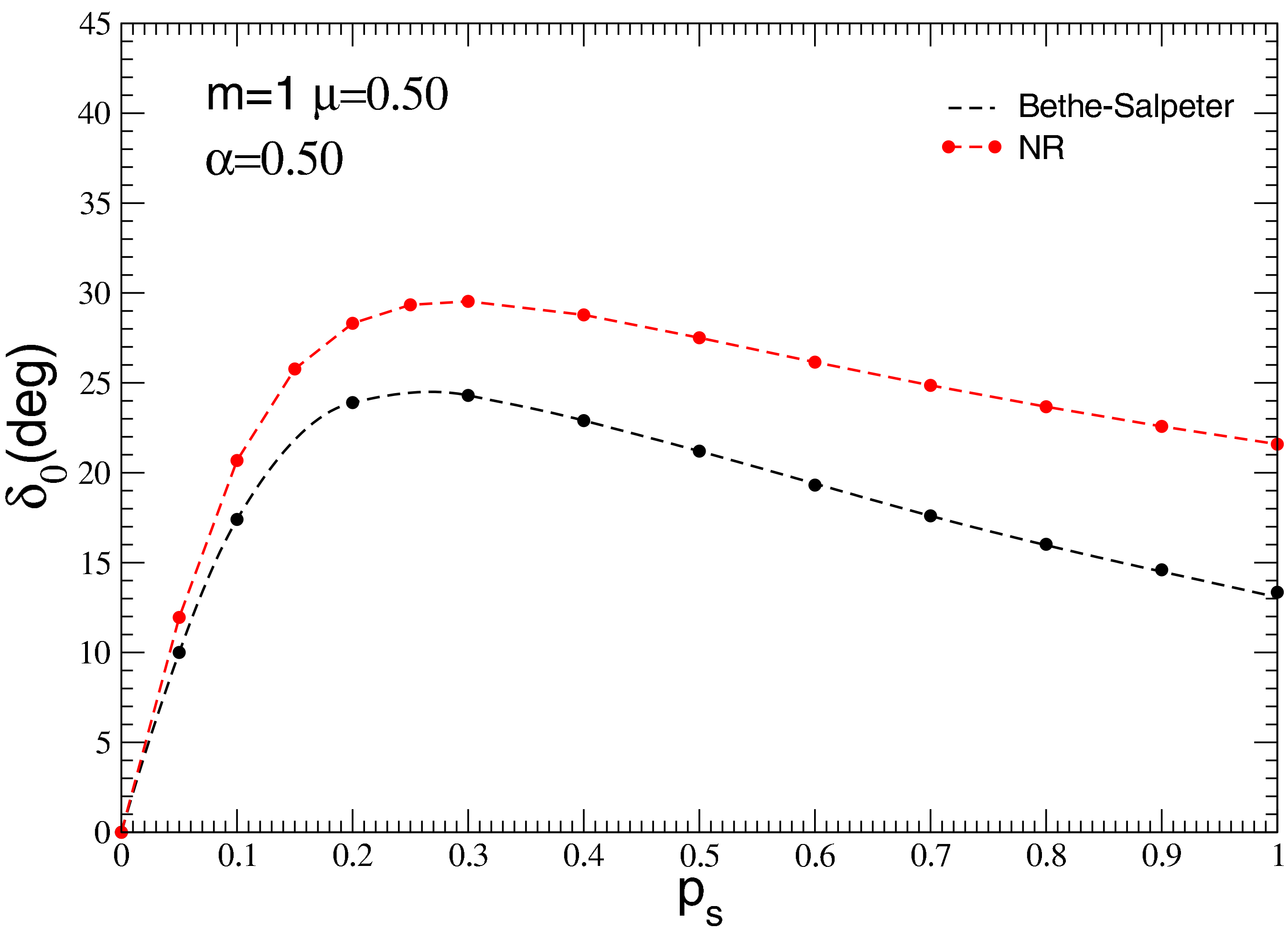
If $s < (2m + \mu)^2$, δ_l must be real. That is $|S_l| = 1$.

• Tests

- Our solution provides the real phase shifts (non-trivial !)
- We independently solved the the BS solution in Euclidean space (and we confirmed Tjon et al.).

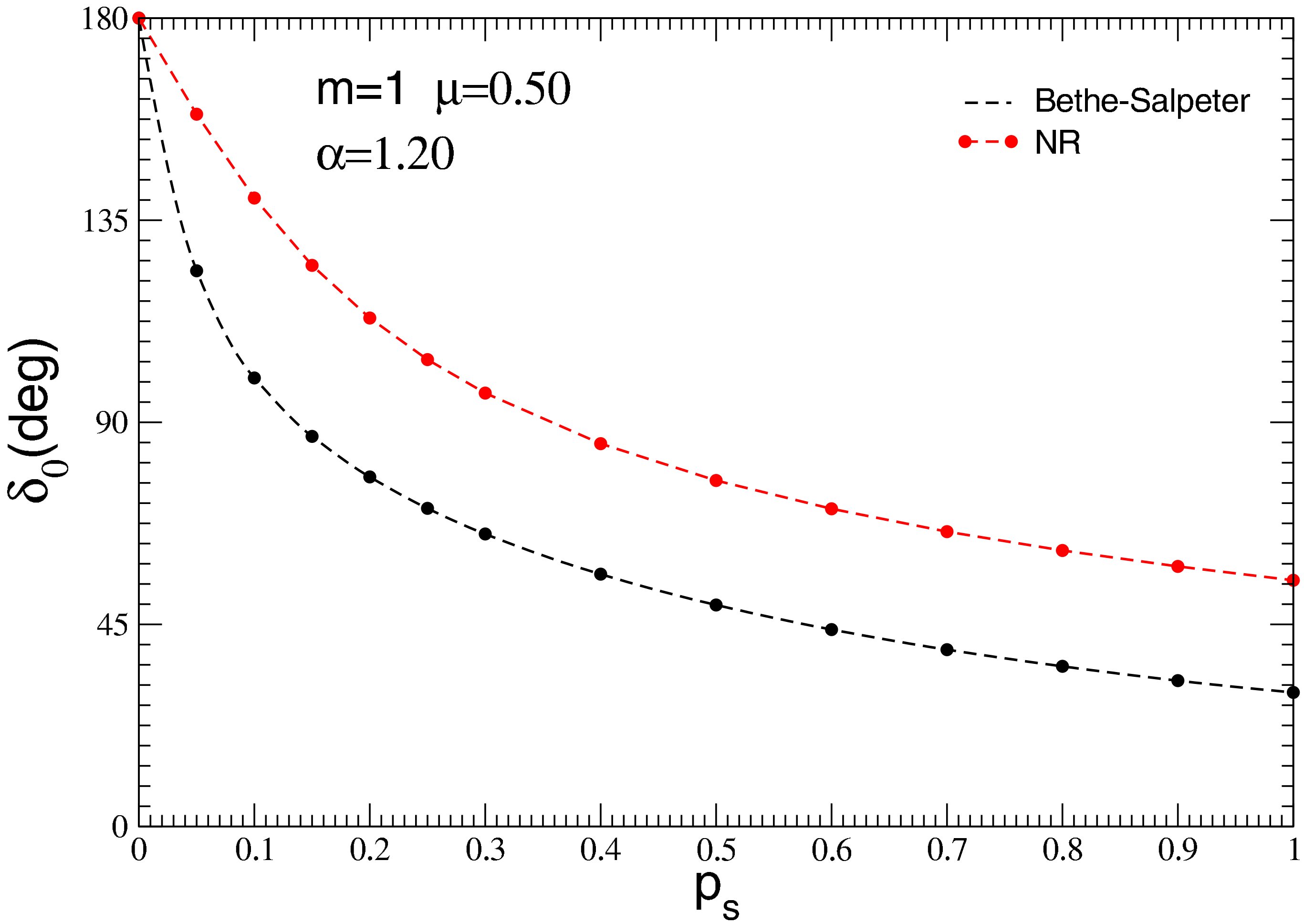
Our Minkowski space solution provides the phase shifts coinciding with ones found via the Euclidean space.

• Phase shift for $\alpha = \frac{g^2}{16\pi m^2} = 0.5$



Precision is better than 0.1%.

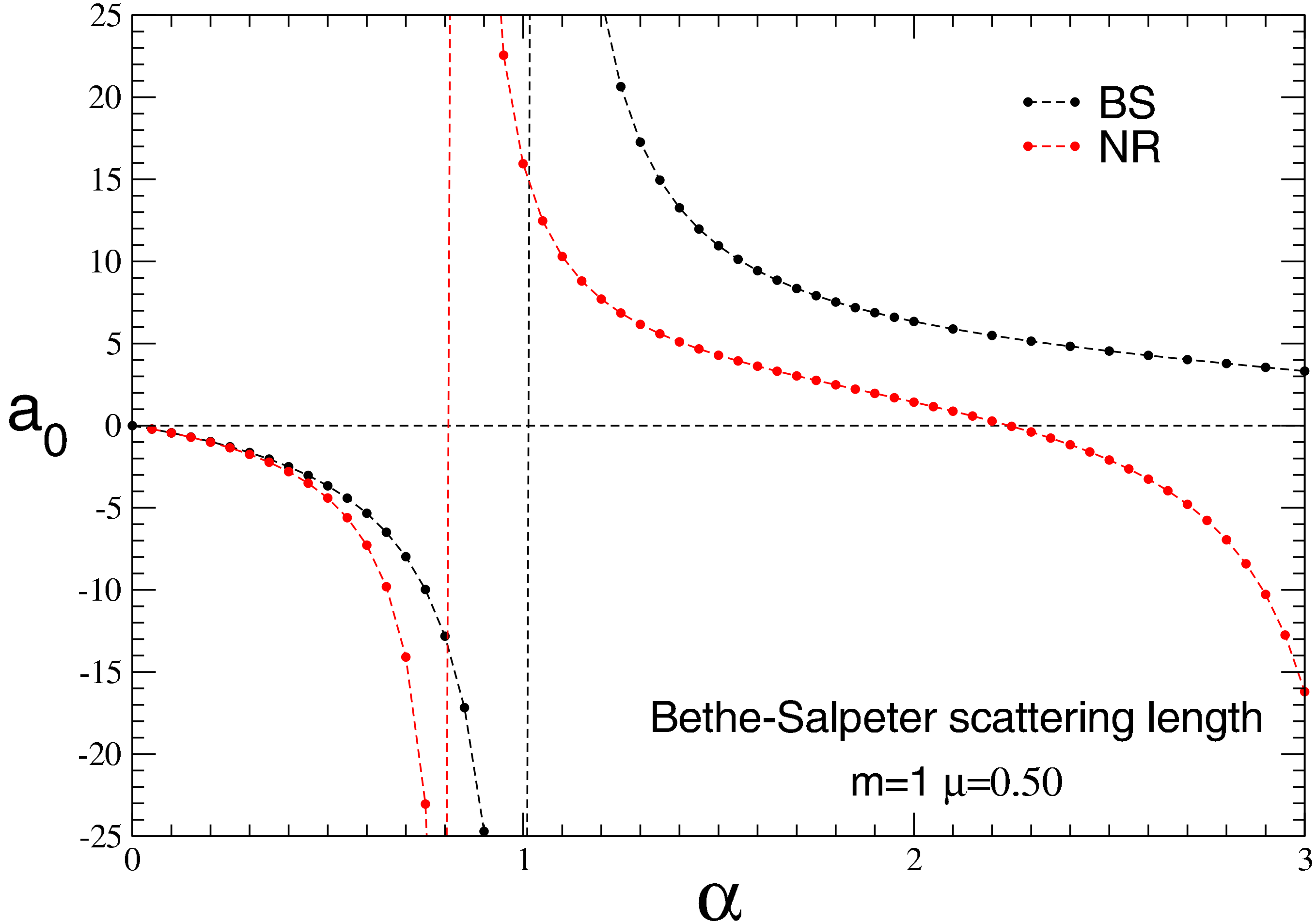
• Phase shift for $\alpha = 1.2$ (bound state)



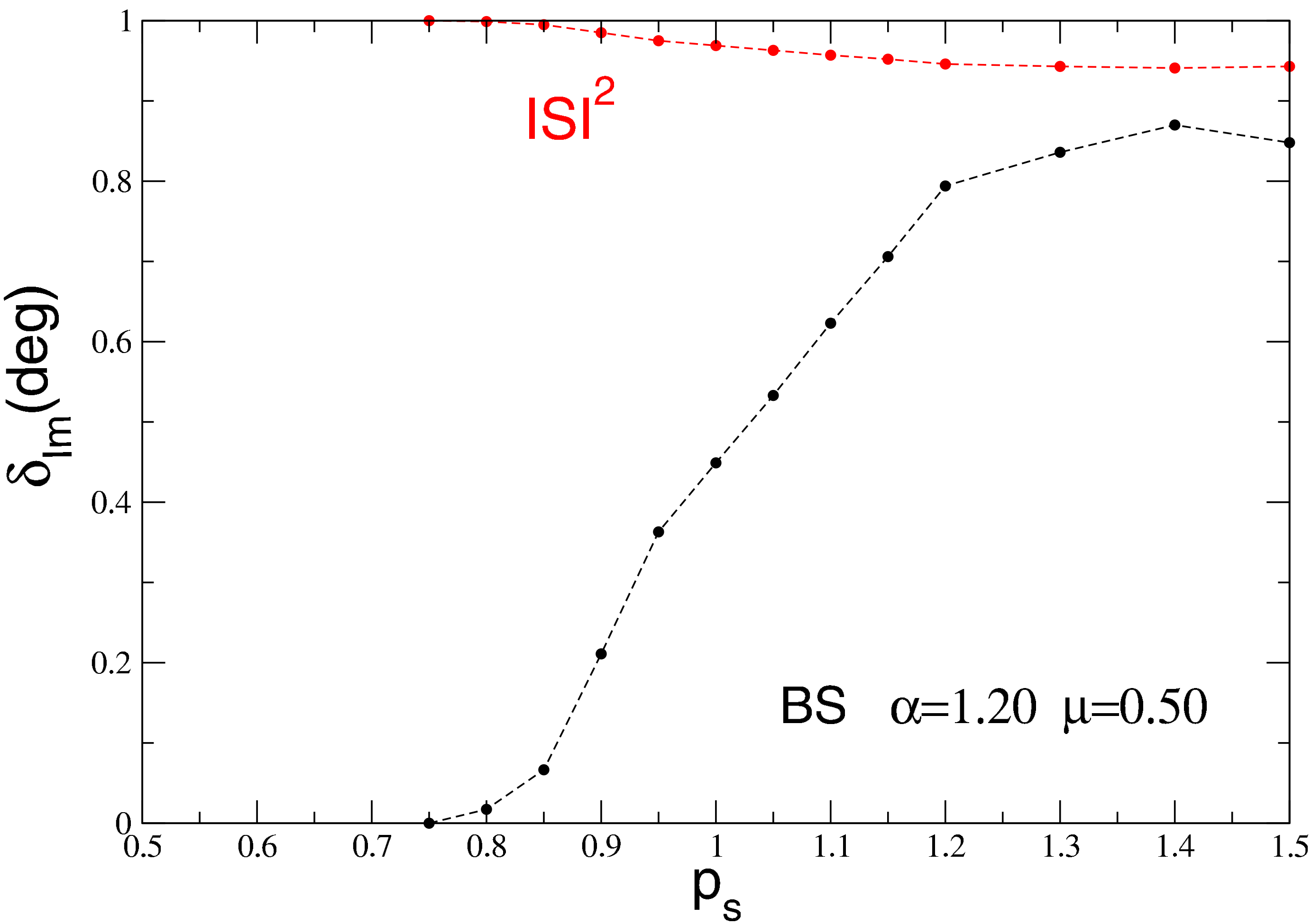
Precision is better than 0.1%.

• Scattering length vs. α

Zero incident energy!

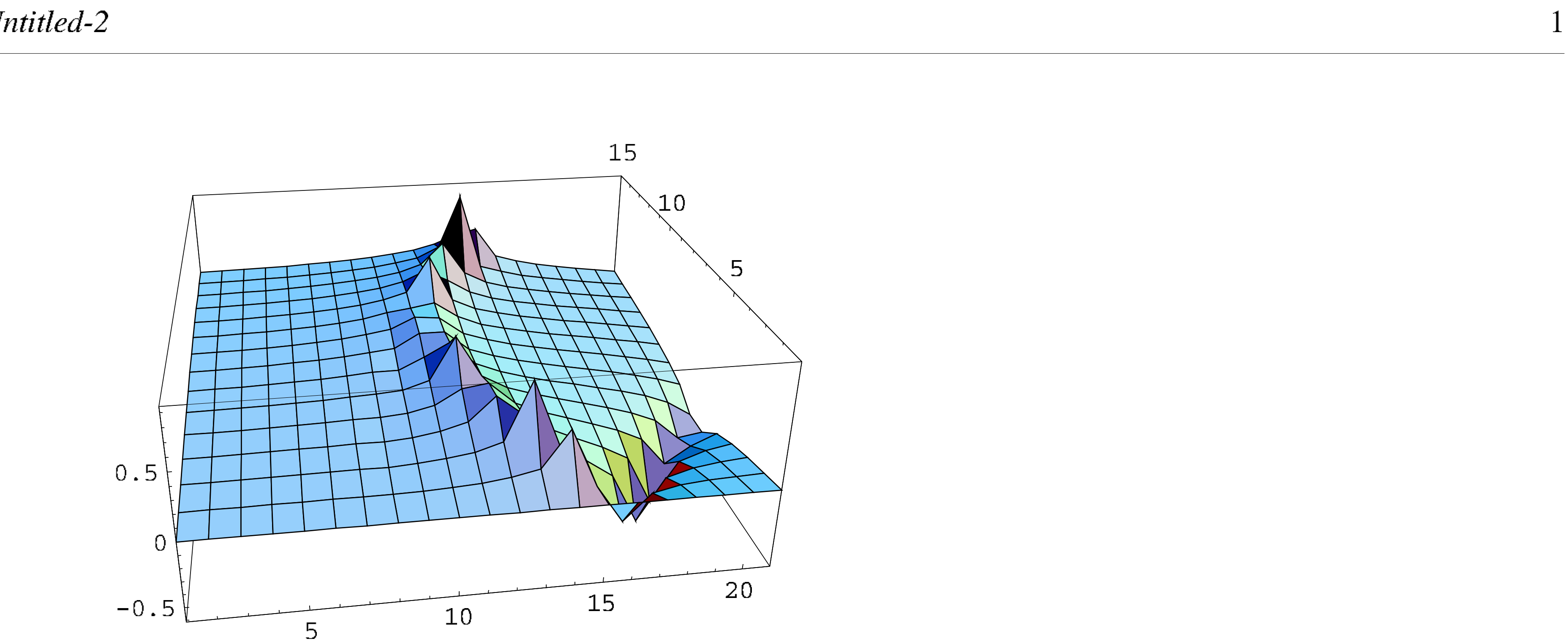


• Inelasticity



$$\mu = 0.5 \rightarrow p''_{threshold} = 0.75$$

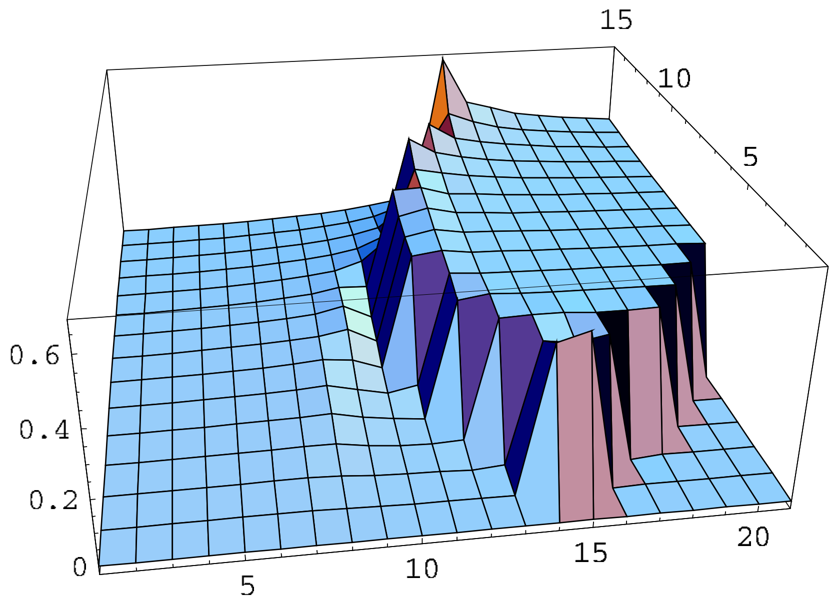
• Real part of off-shell amplitude



$Re[F_0(p_0, p; p'')]$ vs. p, p_0 at $p'' = 0.5$, $\alpha = 0.5$, $\mu = 0.5$.

Imaginary part of off-shell amplitude

Imaginary Part



$Im[F(p_0, p; p'')]$ vs. p, p_0 at $p'' = 0.5$, $\alpha = 0.5$, $\mu = 0.5$.

For the present, the numerical calculations are carried out for the S-wave only.

However, no need in the partial wave decomposition.

The corresponding equations are derived.

Their solution is in progress.

• Conclusions

- The BS equation for the scattering states, for the ladder kernel, is solved in Minkowski space.
  The off-mass-shell BS amplitude is calculated for the first time.
- It is needed to calculate the transition form factor and as an input for the three-body BS-Faddeev equations.
- The relativistic phase shifts and the scattering length considerably differ from the non-relativistic ones.
- Two-body amplitude above threshold and corresponding inelasticity are found.