Stationary Schroedinger Equation and Functional Integral

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- **Feynman path approach**

$$\Psi(t) = \int \frac{D\eta}{C} e^{\frac{i}{\hbar} S_{t,t_0}[\eta]} \Psi(t_0) \quad \Rightarrow \quad i\hbar \frac{d}{dt}\Psi(t) = H\Psi(t)$$

  - ⋆ **Energy is not defined.**

- **Functional integration $\Longrightarrow$ method of solution of second order differential equations of the type**
  $bfi\frac{d}{dt}\Psi(t) = H\Psi(t) \Rightarrow$ Energy is not fixed.

- **Stationary problems. Energy is fixed.**

  $H\Psi_E = E\Psi_E \Rightarrow$ **FI representation** ?

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$$L_x = -\frac{1}{2}\frac{d^2}{dx^2} + W(x).$$

	Equation	Formal solution
1	$i\frac{d}{dt}\Psi = L_x\Psi,$	$\Psi = e^{-itL_x}\Psi_0 = U_t\Psi_0$
2	$L_x\Psi = J,$	$\Psi = \frac{1}{L_x}J = \int\limits_0^\infty dv\ e^{-vL}J = \int\limits_0^\infty dv\ U_{-iv}J$
3	$L_x\Psi = 0,$	$\Psi = \delta(L_x)u_0 = \int\limits_{-\infty}^{\infty}\frac{dt}{2\pi}\ e^{-itL_x}u_0 = \int\limits_{-\infty}^{\infty}\frac{dt}{2\pi}\ U_t u_0$

$$U_t = e^{-itL_x} \quad \rightarrow \quad U_z = e^{-izL_x}, \qquad z = t + iv$$

Homogeneous equation $L_x u(x) = 0$

$$U_z(x) = U_z \delta(x) = e^{-izL_x} \delta(x), \qquad z = t + iv \in C$$

$$L_x U_z(x) = i \frac{d}{dz} U_z(x)$$

$$u(x) = \int_{z_-}^{z_+} dz\; U_z(x), \quad \text{Contour :} \qquad \Gamma = \{z_- < z < z_+ \;\in C\}$$

$$L_x u(x) = i \int_{z_-}^{z_+} dz\; \frac{d}{dz} U_z(x) = i[U_{z_+}(x) - U_{z_-}(x)] = 0$$

Problem is to find

- analytical properties of $U_z(x)$ in the complex plane $z \in C$
- two contours $\Gamma \in C$ with $U_{z_+}(x) = U_{z_-}(x) = 0$.

Stationary Schroedinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]\Psi(x)=E\Psi(x),\qquad L_x\Psi(x)=0$$

$$L_x=H_x-E=\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}-T_E(x)\right],\qquad T_E(x)=E-V(x)$$

.

$$U_z(x)=e^{-i\frac{z}{\hbar}L_x}\delta(x),\qquad L_xU_z(x)=i\hbar\frac{d}{dz}U_z(x)$$

.

$$\Psi_\Gamma(x)=\int_\Gamma dz\;U_z(x),\qquad \Gamma=\{z=t+iv\in \mathbf{C}\}$$

$$U_z(x)=e^{-i\frac{z}{\hbar}L_x}\delta(x)=$$

$$=\sqrt{\frac{m}{2i\pi\hbar}}\cdot\frac{e^{\frac{i}{\hbar}\left(\frac{mx^2}{2z}+Ez\right)}}{\sqrt{z}}\int\limits_{\eta(0)=\eta(1)=0}\frac{D\eta}{C}e^{\frac{i}{\hbar}\int\limits_0^1 d\tau\left[\frac{m}{2}\dot{\eta}^2(\tau)-zV\left(x\tau+\sqrt{z}\;\eta(\tau)\right)\right]}$$

Semiclassical approximation $\hbar \to 0$

$$\Psi(x) = \int\limits_{\Gamma} \frac{dz}{\sqrt{z}} e^{\frac{i}{\hbar}Ez} \int\limits_{\chi(0)=0,\chi(z)=x} \frac{D\chi}{C} e^{\frac{i}{\hbar}\int\limits_0^z d\tau\left[\frac{m\dot{\chi}^2(\tau)}{2} - V(\chi(\tau))\right]}$$

$$\text{Saddle point method over} \begin{cases} \text{functional variable } \chi, \\ \text{complex variable } z \end{cases}$$

$$\Psi_\pm(x) = \frac{C_\pm}{\left(e^{\mp i\frac{\pi}{3}}\kappa^2(x) + p^2(x)\right)^{\frac{1}{4}}} e^{\pm\frac{i}{\hbar}\int^x dx'\sqrt{e^{\mp i\frac{\pi}{3}}\kappa^2(x')+p^2(x')}}, \quad p^2(x) > 0$$

$$\Psi_\pm(x) = \frac{C_\pm}{\left(\kappa^2(x) + |p^2(x)|\right)^{\frac{1}{4}}} e^{\pm\frac{1}{\hbar}\int^x dx'\sqrt{\kappa^2(x')+|p^2(x')|}}, \quad p^2(x) < 0$$

$$\kappa(x) = \left(\frac{mV'(x)\hbar}{2}\right)^{\frac{1}{3}}.$$

Elastic scattering

$$\Psi(\mathbf{x}) = e^{i\mathbf{kx}} + \Phi(\mathbf{x}), \quad \Phi(\mathbf{x}) \to f(k,\theta)\frac{e^{ikr}}{r}$$

,

$$\left(-\frac{1}{2m}\frac{d^2}{d\mathbf{x}^2} + V(\mathbf{x}) - \frac{\mathbf{k}^2}{2m}\right)\Phi(\mathbf{x}) = -V(\mathbf{x})e^{i\mathbf{kx}}$$

$$\Phi(\mathbf{x}) = -\frac{1}{-\frac{1}{2m}\frac{d^2}{d\mathbf{x}^2} + V(\mathbf{x}) - \frac{\mathbf{k}^2}{2m} - i0}V(\mathbf{x})e^{i\mathbf{kx}} = -i\int d\mathbf{y}V(\mathbf{y})\,e^{i\mathbf{ky}}I(\mathbf{y},\mathbf{x},\mathbf{k})$$

$$I(\mathbf{y},\mathbf{x},\mathbf{k}) = \int\limits_0^\infty \frac{m^{\frac{3}{2}}dt}{(2\pi it)^{\frac{3}{2}}}\, e^{\frac{i}{2}\left(\frac{t\mathbf{k}^2}{m} + \frac{(\mathbf{x}-\mathbf{y})^2 m}{t}\right)} \int \frac{D\boldsymbol{\xi}}{C} e^{i\int\limits_0^t d\tau\left[\frac{m}{2}\dot{\boldsymbol{\xi}}^2(\tau) - V\left(\frac{\mathbf{x}}{t}\tau + \left(1-\frac{\tau}{t}\right)\mathbf{y} - \boldsymbol{\xi}(\tau')\right)\right]}$$

$$\xi(0) = \xi(t) = 0.$$

Scattering amplitude

$$f(k,\theta) = \frac{m}{2\pi} \int d\mathbf{y} V(\mathbf{y})\, e^{i\mathbf{q}\mathbf{y}} \int \frac{D\boldsymbol{\xi}}{C} e^{i\int\limits_0^\infty d\tau \left[\frac{m}{2}\dot{\boldsymbol{\xi}}^2(\tau) - V\left(\mathbf{v}_{out}\tau + \mathbf{y} - \boldsymbol{\xi}(\tau)\right)\right]}$$

,

$$\boldsymbol{\xi}(0) = \boldsymbol{\xi}(\infty) = 0$$

$$\mathbf{q} = \mathbf{k}_{in} - \mathbf{k}_{out}, \qquad \mathbf{v}_{out} = \frac{\mathbf{k}_{out}}{m} = \frac{k\mathbf{n}}{m}, \qquad \mathbf{q}^2 = 4k^2 \sin^2\frac{\theta}{2}.$$

Conclusion

- The path integral representation for general solutions of homogeneous second order differential equations is formulated.
- The method is tested on well known examples for wich the exact solution is known.
- This method is implied to find general solutions of the stationary Schroedinger equation in the non-relativistic quantum mechanics.
- The semiclassical approach, which is finite at the turning points, is obtained. The transmission probability through one-dimensional barrier is calculated.
- The path integral representation for elastic potential scattering amplitude is found.