Determination of mass spectrum and decay constants mesons consisting of the c and b quarks

M.Dineykhan, S.A. Zhaugasheva and N.Khabyl

Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna;
al-Farabi Kazakh National University, Almaty, Kazakhstan

Bound states systems:

1.) Quasi-potential equations;

Logunov A.A., Tavkhelidze A.N. Nuovo Cim. 1963. V.29. P.380;
Efimov G.V.Few-Body Syst. 2003. V.33. P.199.
Faustov R.N., Galkin V.O., Mishurov A.Yu. Phys.Lett. B. 1995. V.356. P.516; Phys.
Rev. D. 1996. V.53. P.6302

2.) Nonrelativistic quantum mechanics (NQM).

Phenomenological quark potential models
Quigg C., Rosner J.L. Phys. Rev. 1990.V. 56. P.167
Godfrey S., Isgur N. Phys. Rev. D. 1985. V.32. P.189;
Isgur N., Wise M. Phys. Lett. B. 1984. V. 232.P. 113; 1990. V. 237. P.527.

3.) There is another approach in the framework of the latter direction which is based on the investigation of the asymptotic behavior of the vacuum averaging (of Green's functions) of the scalar charged particle currents in the external gauge field. This functional integral representations can be used for obtaining of the nonrelativistic SE solution in the Feynman functional integral form with the potential consisting of necessary relativisticcorrections.

1.Bound states in the functional approach

Let us determine the mass of a bound state by investigating the asymptotic behavior of the polarization loop function for the charged scalar particle in the external gauge field.

$$\Pi(x-y) = \langle G_{m_1}(x,y|A) G_{m_2}(y,x|A) \rangle_A . \quad (1.1)$$

The Green function Gm(y; xjA) for the scalar particle in the external gauge field is determined from the equation

$$\left[\left(i\frac{\partial}{\partial x_\alpha} + \frac{g}{c\hbar} \cdot A_\alpha(x) \right)^2 + \frac{c^2 m^2}{\hbar^2} \right] G_m(x,y|A) = \delta(x-y) \quad (1.2)$$

When averaging over the external gauge field A(x); let us consider only the lowest order or only the two-point Gauss correlator

$$< \exp\left\{ i \int dx A_\alpha(x) J_\alpha(x) \right\} >_A = \exp\left\{ -\frac{1}{2} \int \int dx dy J_\alpha(x) D_{\alpha\beta}(x-y) J_\beta(y) \right\} \quad (1.3)$$

where J(x) is the real current. The propagator of the gauge field has the following form:

$$D_{\alpha\beta}(x-y) = < A_\alpha(x) A_\beta(y) >_A \quad (1.4)$$

So the external field exists only in a virtual state. The mass of the bound state is usually defined through the loop function in the following way:

$$M = - \lim_{|x-y| \to \infty} \frac{\ln \Pi(x-y)}{|x-y|} \tag{1.5}$$

The solution of (1.2) can be represented as a functional integral in the following way :

$$G_m(x,y|A) = \int_0^\infty \frac{ds}{(4s\pi)^2} \exp\left\{ -sm^2 - \frac{(x-y)^2}{4s} \right\} \int d\sigma_\beta \exp\left\{ ig \int_0^1 d\xi \frac{\partial Z_\alpha(\xi)}{\partial \xi} A_\alpha(\xi) \right\} \tag{1.6}$$

where the following notation is used:

$$Z_\alpha(\xi) = (x-y)_\alpha \xi + y_\alpha - 2\sqrt{s} B_\alpha(\xi) \tag{1.7}$$

$$d\sigma_\beta = N \delta B_\beta \exp\left\{ -\frac{1}{2} \int_0^1 d\xi B'^2(\xi) \right\}$$

with the normalization $B_\alpha(0) = B_\alpha(1) = 0\ ; \quad \text{and} \int d\sigma_\beta = 1$

where N is the normalization constant and $B'(\xi) = \partial B(\xi)/\partial \xi.$

One can obtain for the loop function

$$\Pi(x) = \iint\limits_0^\infty \frac{d\mu_1 d\mu_2}{(8\pi^2 x)^2} \cdot J(\mu_1, \mu_2) \exp\left\{-\frac{|x|}{2}\left(\frac{m_1^2}{\mu_1} + \mu_1\right) - \frac{|x|}{2}\left(\frac{m_2^2}{\mu_2} + \mu_2\right)\right\} \qquad (1.8)$$

Here

$$\begin{aligned} J(\mu_1, \mu_2) &= N_1 N_2 \iint \delta r_1 \delta r_2 \exp\left\{-\frac{1}{2}\int\limits_0^x d\tau \left[\mu_1 \dot{r}_1^2(\tau) + \mu_2 \dot{r}_2^2(\tau)\right]\right\} \times \\ &\times \exp\left\{-W_{1,1} - W_{2,2} + 2W_{1,2}\right\}, \end{aligned} \qquad (1.9)$$

and

$$W_{i,j} = \frac{g^2}{2}(-1)^{i+j} \int\limits_0^x \int\limits_0^x d\tau_1 d\tau_2 Z'^{(i)}{}_\alpha(\tau_1) D_{\alpha\beta}\left(Z^{(i)}(\tau_1) - Z^{(j)}(\tau_2)\right) Z'^{(j)}{}_\beta(\tau_2) \qquad (1.10)$$

We determine the polarization loop function for two charged scalar particles in the external gauge field with masses m_1, m_2. On the other hand, the functional integral represented in (1.9) is analogous to the Feynman path integral for the motion of two particles with masses μ_1, μ_2 in the nonrelativistic quantum mechanics. The interaction between these particles is described by the expression (1.10) which contains the potential and nonpotential parts, in particular, $W_{1,1}, W_{2,2}$ define nonpotential interactions, and $W_{1,2}$ defines potential interactions of a nonlocal nature. Then the interaction Hamiltonian can be represented in the form

$$H = \frac{1}{2\mu_1}\mathbf{P}_1^2 + \frac{1}{2\mu_2}\mathbf{P}_2^2 + V(\mathbf{r}_1, \mathbf{r}_2) \qquad (1.11)$$

From the SE

$$H\Psi(\mathbf{r}_1,\mathbf{r}_2) = E(\mu_1,\mu_2)\Psi(\mathbf{r}_1,\mathbf{r}_2) \qquad (1.12)$$

the $E(\mu_1,\mu_2)$ - eigenvalues of the Hamiltonian (1.11) can be determined. According to (1.12), the functional integral in (1.9) in the $|x-y| \to \infty$ limit can be represented as:

$$\lim_{|x|\to\infty} J(\mu_1,\mu_2) \Longrightarrow \exp\{-x\cdot E(\mu_1,\mu_2)\} \qquad (1.13)$$

In this approximation the integral in (1.9) is evaluated by the saddle-point technique and, hence, for the bound state mass we obtain

$$M = \frac{1}{2}\min_{\mu_1,\mu_2}\left\{\frac{m_1^2}{\mu_1}+\mu_1+\frac{m_2^2}{\mu_2}+\mu_2+2E(\mu_1,\mu_2)\right\} \qquad (1.14)$$

and for j we get the following system of equations:

$$\mu_j - \frac{m_j^2}{\mu_j} + 2\mu_j\frac{dE(\mu_1,\mu_2)}{d\mu_j} = 0\ ; \qquad j=1,\ 2 \qquad (1.15)$$

We have for the masses $$M = \mu_1+\mu_2+\mu\frac{dE}{d\mu}+E(\mu) \qquad (1.16)$$

and $$\mu_1 = \sqrt{m_1^2 - 2\mu^2\frac{dE}{d\mu}}\ ; \qquad \mu_2 = \sqrt{m_2^2 - 2\mu^2\frac{dE}{d\mu}} \qquad \frac{1}{\mu} = \frac{1}{\mu_1}+\frac{1}{\mu_2}$$

Thus, we can determine the mass and constituent mass of bound state systems taking into account relativistic corrections.

2. The interaction Hamiltonian

The total interaction Hamiltonian of quarks is represented as:

$$H = H_c + H_{spin} \qquad (2.17)$$

where Hc is the central Hamiltonian

$$H_c = \frac{1}{2\mu}\vec{P}^2 + \sigma \cdot r - \frac{4}{3}\frac{\alpha_s}{r} \qquad (2.18)$$

The spin-orbit part:

$$H_{spin} = H_{SS} + H_{LS} + H_{TT} \qquad (2.19)$$

Here H_{SS} – is the spin-spin interaction:

$$H_{SS} = \frac{2}{3\mu_1\,\mu_2}\,(\mathbf{S}_1\mathbf{S}_2)\,\Delta V_{\mathrm{v}} = \frac{32\,\pi\,\alpha_s\,(\mathbf{S}_1\mathbf{S}_2)}{9\mu_1\mu_2}\cdot\delta(\mathbf{r}) \qquad (2.20)$$

and spin-orbit interaction:

$$\begin{aligned} H_{LS} \;=\; & \frac{1}{4\,\mu_1^2\mu_2^2}\frac{1}{r}\Big\{\left[\left((\mu_1+\mu_2)^2+2\mu_1\mu_2\right)(\mathbf{L}\cdot\mathbf{S}_+)+(\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)\right]\frac{\partial}{\partial r}V_{\mathrm{v}} \\ & - \left[(\mu_1^2+\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_+)+(\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)\right]\frac{\partial}{\partial r}V_s\Big\}\;, \end{aligned} \qquad (2.21)$$

the tensor interaction: $H_{TT} = \frac{1}{12\mu_1\mu_2}\left[\frac{1}{r}\frac{\partial}{\partial r}V_{\mathrm{v}} - \frac{\partial^2}{\partial r^2}V_{\mathrm{v}}\right] S_{12}$ (2.22)

he v Here Vv - is potential that corresponds to the one-gluon exchange:

$$V_{\mathrm{v}} = -\frac{4\alpha_s}{3}\frac{1}{r} \tag{2.23}$$

and Vs - is the confinement potential

$$V_s = r\sigma \tag{2.24}$$

where:

$$\mathbf{S}_+ = \mathbf{S}_1 + \mathbf{S}_2 \; ; \quad \mathbf{S}_- = \mathbf{S}_1 - \mathbf{S}_2 \; ;$$

$$S_{12} = \frac{4}{(2\ell+3)(2\ell-1)}\left[\mathbf{L}^2\mathbf{S}^2 - \frac{3}{2}(\mathbf{LS}) - 3(\mathbf{LS})^2\right] \tag{2.25}$$

2.1 Determination of the energy spectrum

Now, we proceed to the determination of the quarkonium energy spectrum. The SE

$$H\Psi = E\Psi \tag{2.26}$$

we will solve using the oscillator representation method (OR).
Dineykhan M. et al. Oscillator Representation in Quantum Physics. Lecture Notes in Physics. Berlin: Springer-Verlag. 1995. V. 26.
According to OR method the variables should be transformed in the following way:

$$r = q^{2\rho}, \qquad \Psi \Rightarrow \Psi(q^2) = q^{2\rho\ell}\Phi(q^2) \tag{2.27}$$

As a result we get the modified SE in a d-dimensional auxillary space R^d. The orbital quantum number ` enters the definition of the space dimension d. In the modified SE

$$H\Phi(q) = \varepsilon(E)\ \Phi(q) \tag{2.28}$$

the energy spectrum $\varepsilon(E)$ in R^d equals zero:

$$\varepsilon(E) = 0 \tag{2.29}$$

The interaction Hamiltonian is represented in a correct form over the creation a^+ and annihilation a operators: $H = H_0 + \varepsilon_0(E) + H_I$ (2.30)

Here H_0 is a free oscillators Hamiltonian:

$$H_0 = \omega(a_j^+ a_j) \tag{2.31}$$

and ε_0 is the ground state energy in a zero approximation of OR:

$$\begin{aligned}\varepsilon_0(E) \;=\; & \frac{d\omega}{4} - \frac{4\rho^2 E\ \mu}{\omega^{2\rho-1}} \frac{\Gamma(d/2+2\rho-1)}{\Gamma(d/2)} - \frac{16\alpha_s\mu\rho^2}{3\omega^{\rho-1}} \frac{\Gamma(d/2+\rho-1)}{\Gamma(d/2)} + \\ + \; & \frac{4\rho^2\sigma\ \mu}{\omega^{3\rho-1}} \cdot \frac{\Gamma(d/2+3\rho-1)}{\Gamma(d/2)} + \frac{32\alpha_s\mu\rho}{9\mu_1\mu_2} \cdot \frac{(\vec{S}_1\vec{S}_2)\ \omega^{d/2}}{\Gamma(d/2)} \cdot \delta_{\ell,0} - \\ - \; & \frac{\rho^2\sigma\ \mu}{M_1^2\ \omega^{\rho-1}} \cdot \frac{\Gamma(d/2+\rho-1)}{\Gamma(d/2)} + \frac{4\alpha_s\mu\rho^2 S_{12}}{3\ \mu_1\mu_2} \cdot \frac{\omega^{\rho+1}\Gamma(d/2-\rho-1)}{\Gamma(d/2)} + \\ + \; & \frac{4\alpha_s\mu\rho^2}{3\ M_2^2} \cdot \frac{\omega^{\rho+1}\Gamma(d/2-\rho-1)}{\Gamma(d/2)}\ .\end{aligned} \tag{2.32}$$

We used the following notations:

$$\begin{aligned}\frac{1}{M_1^2} &= \frac{1}{\mu_1^2\ \mu_2^2}\left[(\mu_1^2+\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_+) + (\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)\right]\ ; \\ \frac{1}{M_2^2} &= \frac{1}{\mu_1^2\ \mu_2^2}\left[\left((\mu_1+\mu_2)^2+2\mu_1\mu_2\right)(\mathbf{L}\cdot\mathbf{S}_+) + (\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)\right]\end{aligned} \tag{2.33}$$

The interaction Hamiltonian HI is also represented in the correct form over the creation a+ and annihilation a operators and doesn't contain the quadratic terms of the canonical variables:

$$H_I = \int_0^\infty dx \int \left(\frac{d\eta}{\sqrt{\pi}}\right)^d \exp\left\{-\eta^2(1+x)\right\} : e_2^{-i\sqrt{x\omega}(q\eta)} :$$
$$\left[-\frac{4\rho^2\mu}{\omega^{2\rho-1}}\frac{Ex^{-2\rho}}{\Gamma(1-2\rho)} + \frac{4\rho^2\mu}{\omega^{3\rho-1}}\frac{\sigma x^{-3\rho}}{\Gamma(1-3\rho)} - \frac{16\alpha_s\mu\rho^2}{3\omega^{\rho-1}}\frac{x^{-\rho}}{\Gamma(1-\rho)} - \right.$$
$$- \frac{\sigma\rho^2\mu}{M_1^2\omega^{\rho-1}}\frac{x^{-\rho}}{\Gamma(1-\rho)} + \frac{4\rho^2\mu\alpha_s S_{12}}{3\mu_1\mu_2}\frac{\omega^{\rho+1}x^{\rho}}{\Gamma(1+\rho)} + \frac{4\rho^2\mu\alpha_s}{3M_2^2}\frac{\omega^{\rho+1}x^{\rho}}{\Gamma(1+\rho)} +$$
$$\left. + \frac{16\rho^2\mu\alpha_s(\vec{S}_1\vec{S}_2)}{9\pi\mu_1\mu_2}\lim_{\Lambda\to\infty}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}\frac{\Lambda^{2j+3}}{2j+3}\frac{x^{-2\rho-2\rho j}}{\omega^{2\rho+2\rho j-1}\Gamma(1-2\rho-2\rho j)}\right] \qquad (2.34)$$

Here $\star|$: is the correct form symbol, we used the notations:

$$e_2^{-x} = e^{-x} - 1 + x - \frac{1}{2}x^2$$

The absence of the quadratic terms of the field operators in the interaction Hamiltonian in QFT is equivalent to the condition:

$$\frac{\partial\varepsilon_0(E)}{\partial\omega} = 0 \qquad (2.35)$$

2.2 Determination of the ground state mass spectrum and WF of the mesons consisting of b and c quarks

We determine the properties mesons taking into account only spin-spin interaction. From (2.33) we have:

$$\varepsilon_0(E) = \frac{d\omega}{4} - \frac{4\rho^2 E\mu}{\omega^{2\rho-1}} \frac{\Gamma(3\rho)}{\Gamma(1+\rho)} - \frac{16\alpha_s \mu \rho^2}{3\omega^{\rho-1}} \frac{\Gamma(2\rho)}{\Gamma(1+\rho)} + \frac{4\rho^2 \sigma \mu}{\omega^{3\rho-1}} \frac{\Gamma(4\rho)}{\Gamma(1+\rho)} + \frac{16\alpha_s \mu \rho \omega^{\rho+1}}{3\mu_1 \mu_2} \frac{[s(s+1)-3/2]}{\Gamma(1+\rho)}, \qquad (2.36)$$

where s - is the spin of the mesons. The oscillator frequency:

$$\omega^{3\rho} - \frac{16\alpha_s \rho^2 \omega^{2\rho} \mu}{3} \frac{\Gamma(2\rho)}{\Gamma(2+\rho)} - \frac{4\rho^2 \mu \sigma \Gamma(4\rho)}{\Gamma(2+\rho)} + \frac{16\alpha_s \rho \mu}{3\mu_1 \mu_2} \frac{\omega^{4\rho}[s(s+1)-3/2]}{\Gamma(1+\rho)} = 0 \qquad (2.37)$$

and for the ground state energy:

$$E = \min_{\rho} \left\{ \frac{\omega^{2\rho}\Gamma(2+\rho)}{8\rho^2\mu\Gamma(3\rho)} - \frac{4\alpha_s\omega^{\rho}\Gamma(2\rho)}{3\Gamma(3\rho)} + \right. \tag{2.38}$$

$$\left. + \quad \frac{\sigma\Gamma(4\rho)}{\omega^{\rho}\Gamma(3\rho)} + \frac{4\alpha_s[s(s+1)-3/2]\omega^{3\rho}}{9\rho\mu_1\mu_2\Gamma(3\rho)} \right\}$$

The mass of the singlet and triplet states are defined from the Eqs.:

$$\begin{aligned} \mu_1 - \frac{m_1^2}{\mu_1} + 2\mu_1\frac{dE}{d\mu_1} = 0 \ ; \\ \mu_2 - \frac{m_2^2}{\mu_2} + 2\mu_2\frac{dE}{d\mu_2} = 0 \ . \end{aligned} \tag{2.39}$$

The experimental fitting of current masses of c and b quarks:

$$\begin{aligned} m_c &= 1.275 \pm 0.025 \ GeV; \\ m_b(\bar{M}\bar{S}) &= 4.18 \pm 0.03 \ GeV; \\ m_b(1S) &= 4.65 \pm 0.03 \ GeV. \end{aligned} \tag{2.40}$$

The coupling constant of the quark-gluon interaction is determined as:

$$\alpha_s = \frac{4\pi}{\beta_0 \ln(\frac{\mu_{12}^2}{\Lambda^2})}; \qquad \beta_0 = 11 - \frac{2}{3} n_f; \qquad \mu_{12} = \frac{2\mu_1 \mu_2}{\mu_1 + \mu_2},$$

where nf is the flavor number, $\Lambda = 0.168 GeV$ is the confinement scale of heavy quarks.

The obtained results are in Table 1. According to (2.40) we take the following values of the current masses of quarks: mc = 1,275 GeV and mb = 4,62 GeV The oscillator frequency and the constituent quark masses are defined from the Eqs. (2.40) and (2.40) with the accuracy $\delta_\mu \sim 7.2 \cdot 10^{-10}$ и $\delta_\omega \sim 1.8 \cdot 10^{-9}$ for the charmonium.

The wave function at the origin :

$$|\Psi_n(0)|^2 = \frac{1}{4\pi} \cdot \frac{(\omega^\rho)^{(3+2\ell)}}{\rho\, \Gamma(3\rho + 2\rho\ell)} \cdot \frac{1}{S_n} \qquad (2.42)$$

where

$$\begin{aligned} S_n \;&=\; \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d}{2} - 2\rho - 1)} \frac{\Gamma(1+n)}{\Gamma(\frac{d}{2} + n)} \cdot \frac{1}{\Gamma(1 - 2\rho)} \times \\ &\times \sum_{k=0}^{n} \frac{1}{\Gamma^2(n-k+1)} \cdot \sum_{k=0}^{k} \frac{(-1)^s}{s!(k-s)!} \times \\ &\times \;\Gamma(2n - 2k + s - 2\rho + 1)\Gamma(\frac{d}{2} + k - s + 2\rho - 1). \end{aligned} \qquad (2.43)$$

In particular $$S_0 = 1; \quad S_1 = 1 - \frac{2\rho(1-2\rho)}{1+\rho+2\rho\ell}. \tag{2.44}$$

The lepton decay consant of vector and pseudoscalar mesons:

$$f_p^{NR} = f_v^{NR} = \sqrt{\frac{12}{M_{p,v}}}|\Psi_{p,v}(0)|, \tag{2.45}$$

where $M_{p,v}$ is a mass of vector and pseudoscalar mesons.
The lepton decay width of vector mesons is defined from:

$$\Gamma(V \to \ell\ell) = \frac{16\pi\alpha_{em}^2 e_Q^2}{M_V^2}|\Psi(0)|^2(1 - \frac{16\alpha_s}{3\pi}) \tag{2.46}$$

where $\alpha_{em} = 1/137$ - is the electromagnetic coupling constant; e_Q - charge of a quark, M_V - is a mass of the vector mesons .

Table 1. *The ground state mass spectrum of the mesons consisting of b and c quarks.*

		$\bar{c}c$	$\bar{b}b$	$\bar{b}c$
$S=0$	m_c GeV	1.275	-	1.275
	m_b GeV	-	4.62	4.62
	α_s	0.30366	0.194679	0.248935
	$\sigma\ GeV^2$	0.195	0.153	0.195
	E GeV	0.413530	0.157253	0.363173
	ρ	0.526448	0.651103	0.46495
	ω^ρ GeV	0.652	1.164	0.648335
	μ_c GeV	1.42862	-	1.51306
	μ_b GeV	-	4.73493	4.68082
	M_{our} MeV	2980.05	9400.04	6.2773
	M_{exp} MeV	2980.3 ± 1.2	-	6277 ± 4
	$\lvert\Psi(0)\rvert^2\ GeV^3$	0.047003	0.196457	0.0525517
	$f_\eta\ GeV$	0.435053	0.500795	0.316955
$S=1$	α_s	0.299085	0.194459	0.247683
	E	0.519023	0.216613	0.412532
	ρ	1.03926	1.24871	1.11493
	ω^ρ GeV	1.4311	3.4511	2.0512
	μ_c GeV	1.47617	-	1.53652
	μ_b GeV	-	4.75281	4.71302
	M_{our} MeV	3096.44	9460.3	6330.71
	M_{exp} MeV	3096.916 ± 0.11	9460.3 ± 0.26	-
	$\lvert\Psi(0)\rvert^2\ GeV^3$	0.1004	0.5973	0.219078
	$f_\eta\ GeV$	0.62372	0.8704	0.644412
	Γ_{our} keV	6.135	1.330	-
	Γ_{exp} keV	5.55 ± 0.14	1.340 ± 0.018	-

2.3 Determination of the mass spectrum of mesons with orbital excitement

In this case for the oscillator frequency we have:

$$\omega^{3\rho} - \omega 2\rho \cdot \frac{16\alpha_s \mu \rho^2}{3} \frac{\Gamma(2\rho + 2\rho\ell)}{\Gamma(2 + \rho + 2\rho\ell)} + \frac{4\rho^2 \mu \sigma \Gamma(4\rho + 2\rho\ell)}{\Gamma(2 + \rho + 2\rho\ell)} = 0 \qquad (2.47)$$

and for the energy:

$$E = \min_{\rho} \left\{ \frac{\omega^{2\rho}\Gamma(2 + \rho + 2\rho\ell)}{8\rho^2 \mu \Gamma(3\rho + 2\rho\ell)} - \frac{\alpha \omega^{\rho} \Gamma(2\rho + 2\rho\ell)}{3\Gamma(3\rho + 2\rho\ell)} + \frac{\sigma}{\omega^{\rho}} \cdot \frac{\Gamma(4\rho + 2\rho\ell)}{\Gamma(3\rho + 2\rho\ell)} \right\} \qquad (2.48)$$

The results are (Table 2 and Table 3):

Table 2. *The charmonium mass spectrum with orbital excitements.*

		$J=\ell-1$ S=1	$J=\ell$ S=1	$J=\ell+1$ S=1	$J=\ell$ S=0
$\ell=1$	α_s	0.3013	0.2981	0.2987	0.2978
	E GeV	0.923955	0.960388	0.976759	0.945799
	ρ	0.808694	0.613677	0.775801	0.230383
	ω^ρ GeV	1.14386	0.618518	0.851913	0.276542
	μ_c GeV	1.45188	1.48592	1.47997	1.48936
	M_{h_c} GeV	3.4955	3.54033	3.55515	3.5266
	$\lvert\Psi(0)\rvert^2\ GeV^3$	0.116538	0.0325412	0.0534605	0.00557
	$f_h\ GeV$	0.632513	0.332113	0.424794	0.13763
$\ell=2$	α_s	0.2987	0.2944	0.2962	0.2936
	E GeV	1.2229	1.22267	1.22909	1.21638
	ρ	0.612313	0.595989	0.366686	1.39076
	ω^ρ GeV	0.595536	0.560571	0.323594	5.59744
	μ_c GeV	1.53846	1.5278	1.50771	1.5371
	M_{h_c} GeV	3.81728	3.8145	3.81501	3.81107
	$\lvert\Psi(0)\rvert^2\ GeV^3$	0.1165385	0.025338	0.0077298	1.34144
	$f_h\ GeV$	0.632505	0.282333	0.155929	2.05519

Table 3. *The bottomonium mass spectrum with orbital excitements.*

		$J=\ell-1$ S=1	$J=\ell$ S=1	$J=\ell+1$ S=1	$J=\ell$ S=0
$\ell=1$	α_s	0.3013	0.2981	0.2987	0.2978
	E GeV	0.923955	0.960388	0.976759	0.945799
	ρ	0.808694	0.613677	0.775801	0.230383
	ω^ρ GeV	1.14386	0.618518	0.851913	0.276542
	μ_c GeV	1.45188	1.48592	1.47997	1.48936
	M_{h_c} GeV	3.4955	3.54033	3.55515	3.5266
	$\lvert\Psi(0)\rvert^2\ GeV^3$	0.116538	0.0325412	0.0534605	0.00557
	$f_h\ GeV$	0.632513	0.332113	0.424794	0.13763
$\ell=2$	α_s	0.2987	0.2944	0.2962	0.2936
	E GeV	1.2229	1.22267	1.22909	1.21638
	ρ	0.612313	0.595989	0.366686	1.39076
	ω^ρ GeV	0.595536	0.560571	0.323594	5.59744
	μ_c GeV	1.53846	1.5278	1.50771	1.5371
	M_{h_c} GeV	3.81728	3.8145	3.81501	3.81107
	$\lvert\Psi(0)\rvert^2\ GeV^3$	0.1165385	0.025338	0.0077298	1.34144
	$f_h\ GeV$	0.632505	0.282333	0.155929	2.05519

2.4 The mass and energy spectrum of the mesons with radial excitement.

The energy spectrum has the form:

$$\varepsilon_n(E) = \varepsilon_0(E) + 2n_2\omega + \langle n_2|H_I^c|n_2\rangle + \langle n_2|H_I^s|n_2\rangle \qquad (2.49)$$

where H_I^c is the central part Hamiltonian, H_I^s – is the spin part.

The interaction Hamiltonian is:

$$H_I = \int_0^\infty dx \int \left(\frac{d\eta}{\sqrt{\pi}}\right)^d e^{-\eta^2(1+x)} : e_2^{2i\sqrt{x\omega}(q\eta)} :$$

$$\left\{ - \frac{4\rho^2\mu}{\omega 2\rho - 1} \frac{Ex^{-2\rho}}{\Gamma(1-2\rho)} + \frac{4\rho^2\mu}{\omega^{3\rho-1}} \frac{\sigma x^{-3\rho}}{\Gamma(1-3\rho)} - \frac{16\alpha_s\mu\rho^2}{3\omega^{\rho-1}} \frac{x^{-\rho}}{\Gamma(1-\rho)} \right. \qquad (2.50)$$

$$\left. + \frac{16\rho^2\mu\alpha_s(\vec{S}_1\vec{S}_2)}{9\pi\mu_1\mu_2} \lim_{\Lambda\to\infty} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} \frac{\Lambda^{2j+3}}{(2j+1)} \frac{x^{-2\rho-2\rho j}}{\omega^{2\rho+2\rho j-1}\Gamma(1-2\rho-2\rho j)} \right\}$$

The energy spectrum with radial excitement:

$$E_{n_2} = \frac{\omega^{2\rho}\Gamma(2+\rho)}{8\rho^2\mu\Gamma(3\rho)}\frac{1+\frac{n_2}{1+\rho}}{1+W_1} - \frac{\sigma\Gamma(4\rho)}{\omega^{\rho}\Gamma(3\rho)}\frac{1+W_3}{1+W_1} + \frac{4\alpha_s[S(S+1)-3/2]\omega^{3\rho}}{9\rho\mu_1\mu_3\Gamma(3\rho)}\frac{1+W_s}{1+W_1} \quad (2.51)$$

The oscillator frequency is defined from

$$\omega^{3\rho} - \frac{16\alpha_s\mu\rho^2\omega^{2\rho}\Gamma(2\rho)}{3\Gamma(2+\rho)}\frac{(2\rho-1)\tilde{W}_2-(\rho-1)\tilde{W}_1}{\tilde{W}_1+(2\rho-1)(1+\frac{4n_2}{1+\rho})} + \frac{4\rho^2\mu\sigma\Gamma(4\rho)}{\Gamma(2+\rho)}\frac{(2\rho-1)\tilde{W}_3-(3\rho)\tilde{W}_1}{\tilde{W}_1+(2\rho-1)(1+\frac{n_2}{1+\rho})} +$$

$$+\frac{16\alpha_s\mu\rho[S(S+1)-3/2]\omega^{4\rho}}{9\mu_1\mu_2\Gamma(2+\rho)}\frac{(2\rho-1)\tilde{W}_s+(1+\rho)\tilde{W}_1}{\tilde{W}_1+(2\rho-1)(1+\frac{n_2}{1+\rho})}. \quad (2.52)$$

The obtained result are represented in Table 4

Table 4. *The mass spectrum of mesons consisting of b and c quarks with radial excitement.*

		$\bar{c}c$	$\bar{b}b$	$\bar{b}c$
$S=0$	α_s	0.2745	0.19027	0.22974
	E GeV	0.939195	0.704855	0.79797
	ρ	0.504507	0.45040495	0.537141
	ω^ρ GeV	0.61426	0.913661	0.732053
	μ_c GeV	1.79312	-	2.01377
	μ_b GeV	-	5.115	4.841
	M_{our} MeV	3.6389	9.99276	0.683353
	M_{exp} MeV	3638.9 ± 1.3	-	-
	$\|\Psi(0)\|^2$ GeV^3	0.0409795	0.161118	0.0649499
	f_η GeV	0.367611	0.439865	0.33772
$S=1$	α_s	0.27479	0.18989	0.22996
	E	1.01391	0.728737	0.836904
	ρ	0.644051	0.452765	0.571577
	ω^ρ GeV	0.629255	0.905397	0.73425
	μ_c GeV	1.7888	-	2.03489
	μ_b GeV	-	5.1501	4.896
	M_{our} MeV	3.71148	10.0233	6.88157
	M_{exp} MeV	3686.109 ± 0.012	10.02326 ± 0.0031	-
	$\|\Psi(0)\|^2$ GeV^3	0.025839	0.15568	0.0604622
	f_η GeV	0.289038	0.43172	0.324705
	Γ_{our} keV	0.691849	0.260597	-
	Γ_{exp} keV	2.35 ± 0.04	0.612 ± 0.011	-

3. The E1 transition

The matrix element of E1 transitions from $(n^{2s+1}J),i$, to $(n'^{2s'+1}J'),f$ states has the form:

$$M(i \to f)_\mu = \delta_{s,s'}(-1)^{s+J+J'+1m'}k\sqrt{(2J+1)(2J'+1)(2\ell+1)(2\ell'+1)} \times \times \begin{pmatrix} J' & 1 & J \\ -M' & \mu & M \end{pmatrix} \begin{pmatrix} \ell' & 1 & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell & s & J \\ J' & 1 & \ell' \end{pmatrix} e_Q \, I_{i,f} \,, \qquad (3.53)$$

where parentheses represent the simple 3j symbol, e_Q - is a quark charge and $I_{i,f}$ – is the radial matrix element of the $i \longrightarrow f$ transition:

$$I_{if} = \int_0^\infty dr \; r^2 \Psi^*_{n'\ell'}(r) r \Psi_{nl}(r) \qquad (3.54)$$

where $\Psi_{i,f}$ – is radial WF of the initial and final states. Then the radiative decay with is:

$$\Gamma(i \to f + \gamma) = \frac{4\alpha_{em} \; e_Q^2}{3}(2J'+1)S^E_{if} \, k^3 \; |I_{i,f}|^2 \qquad (3.55)$$

where k – is momentum of a photon:

$$k = \frac{m_i^2 - m_f^2}{2m_i} \quad (3.56)$$

and m_i, m_f are the initial and final state masses. Statistical factor $S_{if}^E = S_{fi}^E$ is

$$S_{if}^E = \max(\ell, \ell') \begin{Bmatrix} J & 1 & J' \\ \ell' & s & \ell \end{Bmatrix}^2 \quad (3.57)$$

After some simplification we get

$$I_{n_2\ell_2}^{n_1\ell_1} = \frac{\rho_1 + \rho_2}{\sqrt{\rho_1\rho_2}} \frac{(\omega_1\rho_1)^{\frac{3}{2}+\ell_1}(\omega_2\rho_2)^{\frac{3}{2}+\ell_2}}{\sqrt{f_{n_1}f_{n_2}}\sqrt{\Gamma(3\rho_1 + 2\rho_1\ell_1)}} \frac{B_{n_1n_2}}{\sqrt{\Gamma(3\rho_2 + 2\rho_2\ell_2)}} \quad (3.58)$$

and

$$B_{n_1n_2} = A(\rho_1, \rho_2)(-1)^{n_1+n_2} \frac{\partial^{n_1+n_2}}{\partial\alpha^{n_1}\partial\beta^{n_2}} \int_0^\infty \frac{dx \cdot x^{-2\rho}}{\Gamma(1-2\rho)} \times \quad (3.59)$$

$$\times \quad \frac{1}{[(1+x-2x\alpha)(1-\alpha\beta) - 2x\beta(1-2\alpha)^2]^{\frac{d}{2}}}\Bigg|_{\beta,\alpha=0},$$

here $A(\rho_1, \rho_2) = \left(\frac{2\omega_1}{\omega_1+\omega_2}\right)^{\frac{d_1}{4}} \left(\frac{2\omega_2}{\omega_1+\omega_2}\right)^{\frac{d_2}{4}} \frac{2^{\rho_1+\rho_2-1}}{(\omega_1+\omega_2)^{\rho_1+\rho_2-1}}$ (3.60)

This integral at certain values of n1 and n2 is evaluated analytically.

Table 5. The radiactive decay with results

Transition $i \to f$	k MeV	I_{if} GeV^{-1}	$\Gamma_{our}(i \to f)$ keV	$\Gamma_{exp}(i \to f)$ keV
$\chi_{c_0} \to \gamma + J/\psi$	376.3	2.33	139.312	-
$\chi_{c_1} \to \gamma + J/\psi$	416.06	1.73	310.3	295.84
$\chi_{c_2} \to \gamma + J/\psi$	429.12	2.18	450.5	~ 500
$1^3D_1 \to \gamma + 1^1P_0$	308.22	1.78	267.92	~ 299
$1^3D_1 \to \gamma + 1^1P_1$	266.90	3.274	146.9	~ 99
$1^3D_1 \to \gamma + 1^1P_2$	253.13	2.751	3.54	~ 3.88
$\chi_{c_0} \to \gamma + \Upsilon$	410.57	1.422	16.81	-
$\chi_{c_1} \to \gamma + \Upsilon$	422.366	1.57	66.9	-
$\chi_{c_2} \to \gamma + \Upsilon$	442.592	0.6644	22.97	–
$1^3D_1 \to \gamma + 1^1P_0$	269.544	0.1526	0.33	–
$1^3D_1 \to \gamma + 1^1P_1$	257.56	0.135	0.06	–
$1^3D_1 \to \gamma + 1^1P_2$	236.929	0.4988	0.024	–

4.Conclusion

1) On the basis of the investigation of the asymptotic behaviour of the correlation functions of the corresponding field currents with the corresponding quantum numbers the analytic method for the determination of the mass spectrum and decay constants of mesons consisting of c and b quarks with relativistic corrections is proposed.

2) The dependence of the constituent mass of quarks on the current mass and on the orbital radial quantum numbers is analytically derived.

3) The mass and wave functions of the mesons are determined via the eigenvalues of nonrelativistic Hamiltonian in which the kinetic energy term is defined by the constituent mass of the bound state forming particles and the potential energy term is determined by the contributions of every possible type of Feynman diagrams with an exchange of gauge field.

4) In the framework of our approach the mass splitting between the singlet and triplet states is determined and the E1 transition rates in the cc, bb and bc systems are calculated.