Long-Range Rapidity Correlations
in the Model with Independent Emitters

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Long Range Rapidity Correlations

two rapidity intervals separated by a gap

- the event multiplicity in the BACKWARD or FORWARD rapidity windows.

\[ \langle n_B \rangle_{n_F} \equiv f(n_F) \quad \text{– the correlation function (regression)} \]
The **linear** correlation function (linear regression):

\[
\langle n_B \rangle_{n_F} = a^{\text{abs}} + b^{\text{abs}} n_F
\]

\[
\frac{\langle n_B \rangle_{n_F}}{\langle n_B \rangle} = a^{\text{rel}} + b^{\text{rel}} \frac{n_F - \langle n_F \rangle}{\langle n_F \rangle} = a^{\text{rel}} + b^{\text{rel}} \left( \frac{n_F}{\langle n_F \rangle} - 1 \right)
\]

\[b^{\text{rel}} = \frac{\langle n_F \rangle}{\langle n_B \rangle} b^{\text{abs}}, \quad a^{\text{rel}} = \frac{\langle n_B \rangle_{n_F=\langle n_F \rangle}}{\langle n_B \rangle}
\]
For a nonlinear correlation function $\langle n_B \rangle_{n_F} = f(n_F)$ (nonlinear regression), expanding in powers of $[n_F - \langle n_F \rangle]$ we have

$$\langle n_B \rangle_{n_F} = f(n_F) = f_0 + f_1[n_F - \langle n_F \rangle] + f_2[n_F - \langle n_F \rangle]^2 + f_3[n_F - \langle n_F \rangle]^3 + ...$$

$$b_{abs} \equiv \left. \frac{d\langle n_B \rangle_{n_F}}{dn_F} \right|_{n_F=\langle n_F \rangle} = f_1, \quad b_{rel} \equiv \left. \frac{d\langle n_B \rangle_{n_F}/\langle n_B \rangle}{dn_F/\langle n_F \rangle} \right|_{n_F=\langle n_F \rangle} = \frac{\langle n_F \rangle}{\langle n_B \rangle} b_{abs}$$

$$\langle n_B \rangle_{n_F=\langle n_F \rangle} = f(\langle n_F \rangle) = f_0, \quad a_{rel} = \frac{\langle n_B \rangle_{n_F=\langle n_F \rangle}}{\langle n_B \rangle}$$
To exclude the trivial dependence on the lengths of the forward $\Delta y_F$ and backward $\Delta y_B$ rapidity windows we define the correlation coefficient $b^{rel}$ using the scaled variables:

Definition 1:

$$b^{rel} \equiv \left. \frac{d\langle n_B \rangle}{dn_F} \langle n_B \rangle \right|_{n_F=\langle n_F \rangle} = \left. \frac{\langle n_F \rangle d\langle n_B \rangle}{dn_F} \right|_{n_F=\langle n_F \rangle} = \frac{\langle n_F \rangle}{\langle n_B \rangle} b^{abs}$$

where $\langle n_F \rangle$ and $\langle n_B \rangle$ are the mean multiplicities in the forward and backward rapidity windows. The $\langle n_B \rangle_{n_F}$ is the correlation function (regression) - the mean multiplicity in the backward window $\Delta y_B$ as a function of the multiplicity in the forward window $\Delta y_F$. 

[Graph showing correlation functions]
In the framework of the model with independent emitters in paper [1] using methods developed in [2] under some very specific assumptions the following formula for the defined correlation coefficient $b^{rel}$ was obtained:

$$b^{rel} = \frac{\kappa \overline{\mu}_F}{\kappa \overline{\mu}_F + 1}.$$

Here the $\kappa$ is the ratio of two scaled variances:

$$\kappa = \frac{V_N}{V_{\mu_F}}, \quad V_N = \frac{D_N}{\langle N \rangle}, \quad V_{\mu_F} = \frac{D_{\mu_F}}{\bar{\mu}_F},$$

$\langle N \rangle$ and $D_N = \langle N^2 \rangle - \langle N \rangle^2$ - the mean number of emitters and the event-by-event variance of the number of emitters.

$\overline{\mu}_F$ and $D_{\mu_F} = \mu_F^2 - \overline{\mu}_F^2$ - the mean multiplicity produced by one emitter in the forward window and the corresponding variance.

For Poisson distributions $V_N = V_{\mu_F} = 1$ and $\kappa = 1$. Clear that the $\bar{\mu}_F$ is depends on the length of the forward rapidity window. In a first approximation we can assume

$$\bar{\mu}_F = \mu_0 F \Delta y_F$$

where $\mu_0 F$ is the average multiplicity produced by one emitter in the forward window per a unit of rapidity.

$$b_{rel} = \frac{\kappa \mu_0 F \Delta y_F}{\kappa \mu_0 F \Delta y_F + 1}.$$ 

So the multiplicity correlation coefficient $b_{rel}$ even defined for scaled variables nevertheless depends through $\mu_F$ on the length of the forward rapidity window $\Delta y_F$ and does not depend on the length of the backward one $\Delta y_B$.

This is because the regression procedure is being made by the forward window. One can find the physical discussion of this phenomenon in ref.:

For a linear correlation function:

\[
\langle n_B \rangle_{n_F} = a^{abs} + b^{abs} n_F, \quad \frac{\langle n_B \rangle_{n_F}}{\langle n_B \rangle} = a^{rel} + b^{rel} \left( \frac{n_F}{\langle n_F \rangle} - 1 \right)
\]

we have exactly:

\[
b^{abs} = \frac{\langle n_B n_F \rangle - \langle n_B \rangle \langle n_F \rangle}{\langle n_F^2 \rangle - \langle n_F \rangle^2} = \frac{\langle n_B n_F \rangle - \langle n_B \rangle \langle n_F \rangle}{D_{n_F}}, \quad b^{rel} = \frac{\langle n_F \rangle}{\langle n_B \rangle} b^{abs},
\]

\[
a^{rel} = \frac{\langle n_B \rangle_{n_F=\langle n_F \rangle}}{\langle n_B \rangle} = 1
\]

So we can take as

\[
\text{Definition 2 :}
\]

\[
b^{abs} = \frac{\langle n_B n_F \rangle - \langle n_B \rangle \langle n_F \rangle}{\langle n_F^2 \rangle - \langle n_F \rangle^2}, \quad b^{rel} = \frac{\langle n_F \rangle}{\langle n_B \rangle} b^{abs}
\]

Note that for a nonlinear correlation function \( \langle n_B \rangle_{n_F} = f(n_F) \)

Definition 1 \( \neq \) Definition 2
If we use the Definition 2 we can obtain the above formula for $b^{rel}$ at very general assumptions. Because to calculate such defined correlation coefficient we need not to calculate the correlation function $\langle B \rangle_F = f(F)$, but only four averages: $\langle F \rangle$, $\langle B \rangle$, $\langle BF \rangle$ and $\langle F^2 \rangle$.

**Calculation of the correlation coefficient**

Simplified notations:

$\langle B \rangle_F = a + bF$, \hspace{1cm} $F \equiv n_F$, \hspace{1cm} $B \equiv n_B$

$P(B, F)$ - basic, \hspace{1cm} $\sum_{B,F} P(B, F) = 1$, \hspace{1cm} $\langle BF \rangle \equiv \sum_{B,F} BF P(B, F)$

$P(F) = \sum_B P(B, F)$, \hspace{1cm} $\sum_F P(F) = 1$, \hspace{1cm} $\langle F \rangle \equiv \sum_F FP(F) = \sum_{B,F} FP(B, F)$

$P(B) = \sum_F P(B, F)$, \hspace{1cm} $\sum_B P(B) = 1$, \hspace{1cm} $\langle B \rangle \equiv \sum_B BP(B) = \sum_{B,F} BP(B, F)$

$P(B, F) = P(F)P_F(B) \Rightarrow P_F(B) = P(B, F)/P(F)$ \hspace{1cm} $\langle B \rangle_F \equiv \sum_B BP_F(B)$
For independent identical emitters:

\[ P(B, F) = \sum_N w(N) \sum_{B_1, \ldots, B_N} \sum_{F_1, \ldots, F_N} \delta_{B_1+\ldots+B_N} \delta_{F_1+\ldots+F_N} \prod_{i=1}^N p(B_i, F_i) \]

For LRC:

\[ p(B_i, F_i) = p_B(B_i) p_F(F_i) \]

Clear that for identical emitters:

\[ \sum_{F_i} p_F(F_i) = 1 \] , \[ \sum_{F_i} F_i p_F(F_i) = \overline{p}_F \] , \[ \sum_{F_i} F_i^2 p_F(F_i) = \overline{p}_F^2 \]

\[ \sum_{B_i} p_B(B_i) = 1 \] , \[ \sum_{B_i} B_i p_B(B_i) = \overline{p}_B \] , \[ \sum_{B_i} B_i^2 p_B(B_i) = \overline{p}_B^2 \]

We denote also

\[ \sum_N w(N) = 1 \] , \[ \sum_N N w(N) = \langle N \rangle \] , \[ \sum_N N^2 w(N) = \langle N^2 \rangle \]

The variances:

\[ D_N = \langle N^2 \rangle - \langle N \rangle^2 \] , \[ D_{\mu_F} = \overline{\mu}_F^2 - \mu_F^2 \]

and the scaled variances:

\[ V_N = D_N / \langle N \rangle \] , \[ V_{\mu_F} = D_{\mu_F} / \overline{\mu}_F \]
Calculation of \( \langle n_F^2 \rangle \equiv \langle F^2 \rangle \) as an example

\[
\langle n_F^2 \rangle \equiv \langle F^2 \rangle = \sum_F F^2 P(F) = \sum_F F^2 \sum_N w(N) \sum_{F_1, \ldots, F_N} \delta_{F_1+\ldots+F_N} \prod_{i=1}^N p_F(F_i) = 
\]

\[
= \sum_N w(N) \sum_{F_1, \ldots, F_N} (F_1 + \ldots + F_N)^2 \prod_{i=1}^N p_F(F_i) = 
\]

\[
= \sum_N w(N) \sum_{F_1, \ldots, F_N} \left[ \sum_{i=1}^N F_i^2 + \sum_{i \neq j=1}^N F_i F_j \right] \prod_{i=1}^N p_F(F_i) = 
\]

\[
= \sum_N w(N) [N \mu_F^2 + (N^2 - N) \mu_F^2] = \langle N \rangle \mu_F^2 + (\langle N^2 \rangle - \langle N \rangle) \mu_F^2 = 
\]

\[
= \langle N \rangle (\mu_F^2 - \mu_F^2) + \langle N^2 \rangle \mu_F^2
\]

So we find

\[
\langle n_F^2 \rangle = \langle N \rangle D_{\mu_F} + \langle N^2 \rangle \mu_F^2
\]

and

\[
D_{n_F} \equiv \langle n_F^2 \rangle - \langle n_F \rangle^2 = \langle N \rangle D_{\mu_F} + \langle N^2 \rangle \mu_F^2 - \langle N \rangle^2 \mu_F^2 = \langle N \rangle D_{\mu_F} + D_N \mu_F^2
\]
Gathering we find

\[ b_{\text{abs}} \equiv \frac{\langle BF \rangle - \langle B \rangle \langle F \rangle}{\langle F^2 \rangle - \langle F \rangle^2} = \frac{\langle BF \rangle - \langle B \rangle \langle F \rangle}{D_{n_F}} = \frac{D_N \bar{\mu}_B \bar{\mu}_F}{\langle N \rangle D_{\mu_F} + D_N \bar{\mu}_F^2} \]

and

\[ b_{\text{rel}} = \frac{\langle n_F \rangle}{\langle n_B \rangle} b_{\text{abs}} = \frac{\langle N \rangle \bar{\mu}_F}{\langle N \rangle \bar{\mu}_B} b_{\text{abs}} = \frac{\bar{\mu}_F}{\bar{\mu}_B} b_{\text{abs}} = \frac{D_N \bar{\mu}_F^2}{\langle N \rangle D_{\mu_F} + D_N \bar{\mu}_F^2} = \frac{\kappa \bar{\mu}_F}{\kappa \bar{\mu}_F + 1}, \]

where the \( \kappa \) is the ratio of two scaled variances:

\[ \kappa = \frac{V_N}{V_{\mu_F}}, \quad V_N = \frac{D_N}{\langle N \rangle}, \quad V_{\mu_F} = \frac{D_{\mu_F}}{\bar{\mu}_F}, \]

\[ D_N = \langle N^2 \rangle - \langle N \rangle^2, \quad D_{\mu_F} = \bar{\mu}_F^2 - \bar{\mu}_F^2. \]
Comparing the definitions

In the case of nonlinear regression:

\[ \langle n_B \rangle_{n_F} \equiv f(n_F) = f_0 + f_1[n_F - \langle n_F \rangle] + f_2[n_F - \langle n_F \rangle]^2 + f_3[n_F - \langle n_F \rangle]^3 + \ldots \]

**Definition 1 ≠ Definition 2**

By **Definition 1**:

\[ \bar{b}^{abs} = \left. \frac{d\langle n_B \rangle_{n_F}}{dn_F} \right|_{n_F=\langle n_F \rangle} = f_1 \quad \bar{b}^{rel} = \frac{\langle n_F \rangle}{\langle n_B \rangle} \bar{b}^{abs}, \]

By **Definition 2**:

\[ b^{abs} = \frac{\langle n_Bn_F \rangle - \langle n_B \rangle \langle n_F \rangle}{\langle n_F^2 \rangle - \langle n_F \rangle^2} = \frac{\langle n_Bn_F \rangle - \langle n_B \rangle \langle n_F \rangle}{D_{n_F}}, \quad b^{rel} = \frac{\langle n_F \rangle}{\langle n_B \rangle} b^{abs}, \]

\[ b^{abs} - \bar{b}^{abs} = f_2\frac{[n_F - \langle n_F \rangle]^3}{D_{n_F}} + f_3\frac{[n_F - \langle n_F \rangle]^4}{D_{n_F}} + \ldots \]
Coefficient $a_{rel}$

$$a_{rel} = \frac{\langle n_B \rangle_{n_F=\langle n_F \rangle}}{\langle n_B \rangle} = \frac{f(\langle n_F \rangle)}{\langle f(n_F) \rangle}$$

$$\langle n_B \rangle_{n_F=\langle n_F \rangle} \equiv f(n_F) = f_0 + f_1[n_F - \langle n_F \rangle] + f_2[n_F - \langle n_F \rangle]^2 + f_3[n_F - \langle n_F \rangle]^3 + \ldots$$

$$\langle n_B \rangle = \langle f(n_F) \rangle = f_0 + f_2\langle [n_F - \langle n_F \rangle]^2 \rangle + f_3\langle [n_F - \langle n_F \rangle]^3 \rangle + \ldots$$

$$\langle n_B \rangle - \langle n_B \rangle_{n_F=\langle n_F \rangle} = f_2 D_{n_F} + f_3 \langle [n_F - \langle n_F \rangle]^3 \rangle + \ldots$$

So $a_{rel} = 1$ for linear correlation function.

In the next (quadratic) approximation:

If $a_{rel} < 1$ - the correlation function is convex downwards: $f_2 > 0$.

If $a_{rel} > 1$ - the correlation function is convex upwards: $f_2 < 0$. 
Conclusions

The formula for the long-range multiplicity correlation coefficient in the model with independent emitters is obtained at very general assumptions:

\[ b_{rel} = \frac{\kappa \bar{\mu}_F}{\kappa \bar{\mu}_F + 1}, \]

where the \( \kappa \) is the ratio of two scaled variances: \( \kappa = \frac{\langle N \rangle}{\langle N_F \rangle}, \) \( \bar{\mu}_F = \frac{D_{\mu F}}{\bar{\mu}_F} \) and \( \bar{\mu}_F \) - the mean multiplicity produced by one emitter in the forward window.

The multiplicity correlation coefficient defined for scaled variables nevertheless depends through \( \bar{\mu}_F \) on the length of the forward rapidity window \( \Delta y_F \) and does not depend on the length of the backward one \( \Delta y_B \):

\[ b_{rel} = \frac{\kappa \bar{\mu}_0 F \Delta y_F}{\kappa \bar{\mu}_0 F \Delta y_F + 1}, \quad \bar{\mu}_F = \frac{\bar{\mu}_0 F \Delta y_F}{\Delta y_F} \]

where \( \bar{\mu}_0 F \) is the average multiplicity produced by one emitter in the forward window per a unit of rapidity.

This is due to the regression procedure is being made by the forward window.

One can find the physical discussion of this phenomenon in ref.: