The f_2- and ρ_3-mesons from analysis of multi-channel pion-pion scattering

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Outline:

- **Motivation**
- **The S-matrix formalism for N coupled channels**
- **Analysis of the $I^G J^{PC} = 0^+2^{++}$ sector**
- **Analysis of the isovector F-wave of $\pi\pi$ scattering**
- **Discussion and conclusions**

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Motivation

We present results of the coupled-channel analysis of data on processes $\pi\pi \to \pi\pi, K\overline{K}, \eta\eta$ in the $I^G J^{PC} = 0^+2^{++}$ sector and on the $\pi\pi$ scattering in the 1^+3^{--} sector.

$\underline{0^+2^{++}}$: From 13 resonances, discussed in the PDG issue (*C.Amsler et al. (PDG), PL B **667 (2008) 1*), the 9 ones ($f_2(1430)$, $f_2(1565)$, $f_2(1640)$, $f_2(1810)$, $f_2(1910)$, $f_2(2000)$, $f_2(2020)$, $f_2(2150)$, $f_2(2220)$) must be confirmed in various experiments and analyses.**

In analysis of $p\overline{p} \to \pi\pi, \eta\eta, \eta\eta'$ (*V.V.Anisovich et al., IJMP A **20 (2005) 6327*), 5 resonances – $f_2(1920)$, $f_2(2000)$, $f_2(2020)$, $f_2(2240)$ and $f_2(2300)$ – have been obtained, where the $f_2(2000)$) is a candidate for a glueball.**

In our analysis of $\pi\pi \to \pi\pi, K\overline{K}, \eta\eta$ (*SBKN-PRD'2010*) we supported the conclusion about the $f_2(2000)$.

This sector might be interested more because here multi-quark states might be observed as separate states, whereas in the scalar sector they (owing to their very large width) can be manifest themselves only in distortion of the $q\bar{q}$ picture.

$\underline{I^G J^{PC} = 1^+ 3^{--}}$: In our multi-channel analysis (*Yu.S.Surovtsev, P.Bydžovský, Frascati Phys. Series* ***XLVI*** *(2007) 1535; NP A* ***807*** *(2008) 145; SBKN-PRD'2010*) **of the P-wave $\pi\pi$ scattering data (***S.D.Protopopescu et al., PR D* ***7*** *(1973) 1279; B.Hyams et al., NP B* ***64*** *(1973) 134; P.Estabrooks, A.D.Martin, NP B* ***79*** *(1974) 301*) **and in re-analysis of $e^+e^- \to \omega\pi^0$ (***I.Yamauchi, T.Komada, Frascati Phys. Series* ***XLVI*** *(2007) 445*)**, there was confirmed the old issue (***N.M. Budnev et al., PL B* ***70*** *(1977) 365*) **that the 1st ρ-like meson is $\rho(1250)$ unlike $\rho(1450)$. For the $\rho(1250)$ there are the possible SU(3) partners: the isodoublet $K^*(1410)$ and the isoscalar $\omega(1420)$, for which one obtains the mass values in range 1350-1460 MeV in various works (***PDG'08*)**. The GM-O formula gives for the mass of the 8th component of this octet the value about 1460 MeV.**

The result about the $\rho(1250)$ is consistent with predictions of some quark models (*S.B.Gerasimov, A.B.Govorkov, ZP C* ***13* *(1982) 43;* ***29*** *(1985) 61; E. van Beveren, G.Rupp, T.A.Rijken, C.Dullemond, PR D* ***27*** *(1983) 1527*). However, if existence of the $\rho(1250)$ is confirmed, the mainstream quark-potential models, e.g., (*S.Godfrey, N.Isgur, PR D* **32** *(1985) 189*) will require substantial revisions. In these models, the first ρ-like meson is usually predicted by about 200 MeV higher than the $\rho(1250)$, and also the first K^*-like meson is obtained at 1580 MeV, whereas the corresponding well established resonance has the mass of 1410 MeV. To the point, in the isoscalar-scalar and isoscalar-tensor sectors, there are also disagreements with predictions of the indicated model, e.g., with respect to the $f_0(600)$ and $f_0(1500)$ in the scalar sector and to the 2nd $q\bar{q}$ nonet in the tensor sector.**

Therefore, it is important to check if the result on the $\rho(1250)$ is supported by investigation of other mesonic sectors.

Considering in the (J, M^2)-plane the corresponding daughter ρ-trajectory, related to the $\rho(1250)$, one can conclude that there should exist the 1^+3^{--}-state at about 1950 MeV –"$\rho_3(1950)$". Therefore, it is worth testing this by analyzing accessible data on the F-wave $\pi\pi$ scattering (*B.Hyams et al., NP B **64 (1973) 134*).**

In this investigation, we applied the multi-channel S-matrix approach (*SBKN-PRD'2010*). To generate resonance poles and zeros on the Riemann surface, there are used multi-channel Breit–Wigner forms taking into account the Blatt–Weisskopf barrier factors, conditioned by spins of resonances (*J.Blatt, V.Weisskopf, "Theoretical nuclear physics", Wiley, N.Y., 1952*).

The S-matrix formalism for N coupled channels

The N-channel S-matrix is determined on the 2^N-sheeted Riemann surface. The elements S_{ab} $(a, b = 1, 2, \cdots, N$ denote channels) have the right-hand cuts along the real axis of the s complex plane (s is the invariant total energy squared), related to the considered channels and starting with the channel thresholds s_i $(i = 1, \cdots, N)$, and the left-hand cuts related to the crossed channels.
The main model-independent part of resonance contributions is given by poles and zeros on the Riemann surface. Generally this representation of resonances is obtained with the help of formulas of analytic continuations of the matrix elements for the coupled processes to unphysical sheets of the Riemann surface, performed for the N-channel case in (*D.Krupa, V.Meshcheryakov, Yu.Surovtsev, NC A* ***109* *(1996) 281*).**

The Le Couteur–Newton relations (*K.J.LeCouteur, Proc.Roy.Soc. A* ***256*** *(1960) 115; R.G.Newton, J.Math.Phys.* ***2*** *(1961) 188***) are used to generate the resonance poles and zeros on the Riemann surface. These relations express the S-matrix elements of all coupled processes in terms of the Jost matrix determinant $d(k_1, \cdots, k_N)$ ($k_i = \frac{1}{2}\sqrt{s - s_i}$) that is a real analytic function with the only branch-points at $k_i = 0$:**

$$S_{aa} = \frac{d(k_1, \cdots, k_{a-1}, -k_a, k_{a+1}, \cdots, k_N)}{d(k_1, \cdots, k_N)},$$

$$S_{aa}S_{bb} - S_{ab}^2 = \frac{d(k_1, \cdots, k_{a-1}, -k_a, k_{a+1}, \cdots, k_{b-1}, -k_b, k_{b+1}, \cdots, k_N)}{d(k_1, \cdots, k_N)}.$$

The real analyticity implies $d(s^*) = d^*(s)$ **for all** s**.**
The N-channel unitarity requires

$$|d(k_1, \cdots, -k_a, \cdots, k_N)| \leq |d(k_1, \cdots, k_N)|, \;\; a = 1, \cdots, N,$$

$$|d(-k_1, \cdots, -k_a, \cdots, -k_N)| = |d(k_1, \cdots, k_a, \cdots, k_N)|$$

to hold for physical s**-values.**

The d-function is taken as $d = d_B d_{res}$. For the resonance part d_{res} there are used multi-channel Breit–Wigner forms

$$d_{res}(s) = \prod_r \left[M_r^2 - s - i \sum_{i=1}^{N} \rho_{ri}^{2J+1} R_{ri} f_{ri}^2 \right]$$

where $\rho_{ri} = 2k_i/\sqrt{M_r^2 - s_i}$, f_{ri}^2/M_r indicates to the partial width; $R_{ri}(s, M_r, s_i, r_{ri})$ is the Blatt–Weisskopf barrier factors with s_i the channel threshold, r_{ri} a radius of the i-channel decay of the state "r".

The background part d_B is introduced by a natural way: just when some channel is open, in the background there are arisen the corresponding elastic and inelastic phase shifts.

$$d_B = \exp\left[-i \sum_{i=1}^{N} \left(\sqrt{\frac{s - s_i}{s}} \right)^{2J+1} (a_i + i b_i) \right].$$

From the formulas of analytic continuations of the matrix elements for the coupled processes to unphysical sheets of the Riemann surface (*D.Krupa, V.Meshcheryakov, Yu.Surovtsev, NC A* ***109*** *(1996) 281***), one can conclude that only on the sheets with the numbers 2^i $(i = 1, \cdots, N)$, i.e. II, IV, VIII, XVI,..., the analytic continuations have the form $\propto 1/S_{ii}^{\mathrm{I}}$. This means that the pole positions of resonances only on these sheets are at the same points of the s-plane, as the resonance zeros on the physical sheet, and are not shifted due to the coupling of channels. Therefore,** *the resonance parameters should be calculated from the pole positions only on these sheets.*

In the 4-channel cases, considered below, we have the 16-sheeted Riemann surface; to sheets II, IV, VIII, and XVI there correspond the following signs of analytic continuations of the quantities $\mathrm{Im}\sqrt{s-s_1}$, $\mathrm{Im}\sqrt{s-s_2}$, $\mathrm{Im}\sqrt{s-s_3}$, and $\mathrm{Im}\sqrt{s-s_4}$: $-+++, +-++, ++-+,$ and $+++-$, respectively.

Analysis of the $I^G J^{PC} = 0^+ 2^{++}$ sector

When analyzing the isoscalar D-waves of processes $\pi\pi \to \pi\pi, K\overline{K}, \eta\eta$, we considered explicitly also the channel $(2\pi)(2\pi)$ $(i = 4)$. I.e. for the resonance part of the function $d(\sqrt{s-s_1}, \sqrt{s-s_2}, \sqrt{s-s_3}, \sqrt{s-s_4})$ should be applied the 4-channel Breit-Wigner form. The Blatt–Weisskopf barrier factor for a particle with $J = 2$ is

$$R_{ri} = \frac{9 + \frac{3}{4}(\sqrt{M_r^2 - s_i}\; r_{ri})^2 + \frac{1}{16}(\sqrt{M_r^2 - s_i}\; r_{ri})^4}{9 + \frac{3}{4}(\sqrt{s - s_i}\; r_{ri})^2 + \frac{1}{16}(\sqrt{s - s_i}\; r_{ri})^4}$$

with radii of 0.943 fm for all resonances in all channels, except for $f_2(1270)$ and $f_2(1960)$ for which they are: for $f_2(1270)$, 1.498, 0.708 and 0.606 fm in channels $\pi\pi$, $K\overline{K}$ and $\eta\eta$, for $f_2(1960)$, 0.296 fm in channel $K\overline{K}$.

$$d_B = \exp\left[-i\sum_{n=1}^{3}\left(\sqrt{\frac{s-s_n}{s}}\right)^5 (a_n + ib_n)\right].$$

$$a_1 = \alpha_{11} + \frac{s-4m_K^2}{s}\,\alpha_{12}\,\theta(s-4m_K^2) + \frac{s-s_v}{s}\,\alpha_{10}\,\theta(s-s_v)),$$

$$b_n = \beta_n + \frac{s-s_v}{s}\,\gamma_n\,\theta(s-s_v).$$

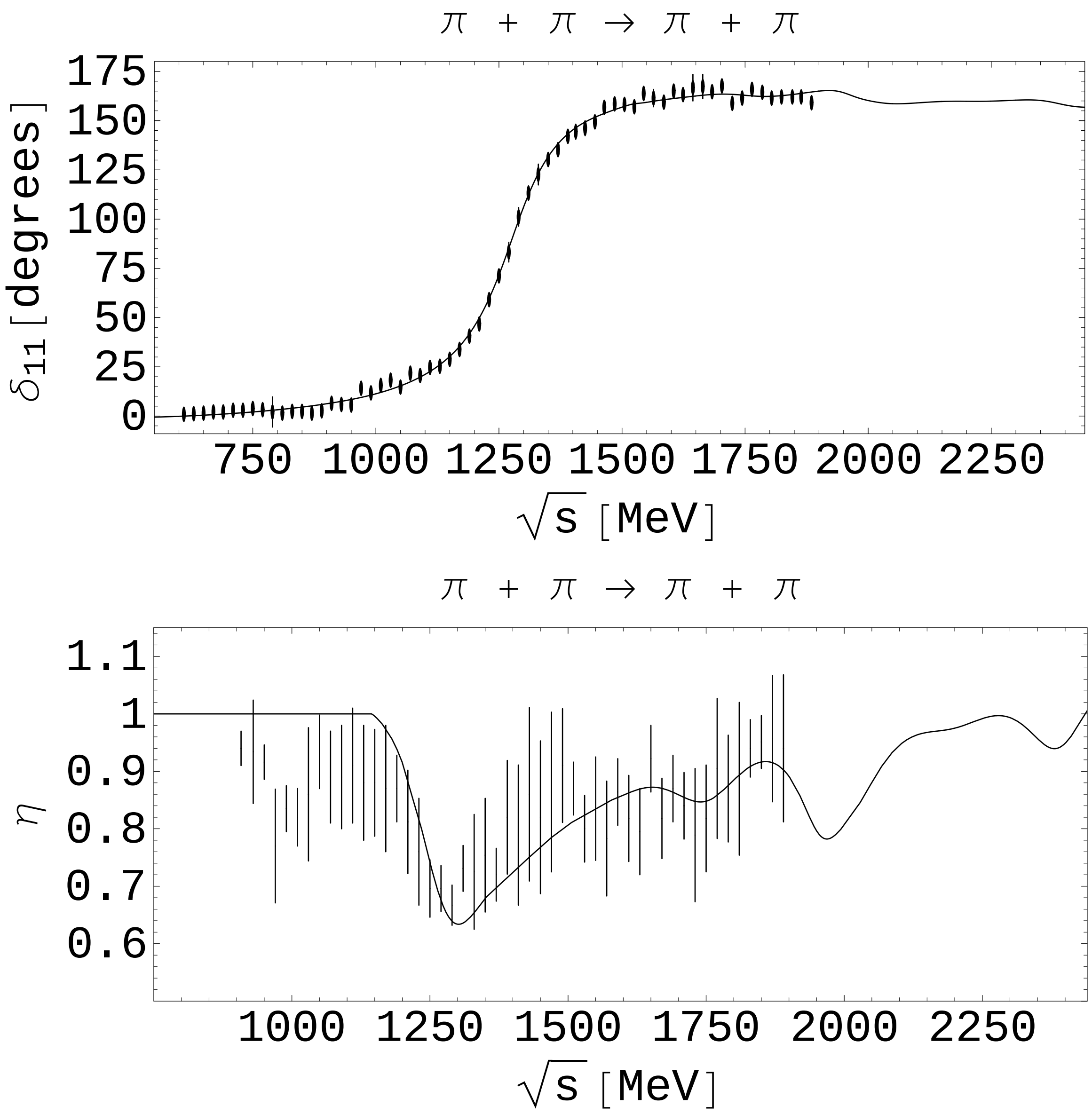
$s_v \approx 2.274$ GeV^2 is the combined threshold of channels $\eta\eta',\ \rho\rho,\ \omega\omega$.

The data for the $\pi\pi$ scattering are taken from an energy-independent analysis by B.Hyams et al. (*NP B* ***64*** *(1973) 134; ibid.* ***100*** *(1975) 205*).

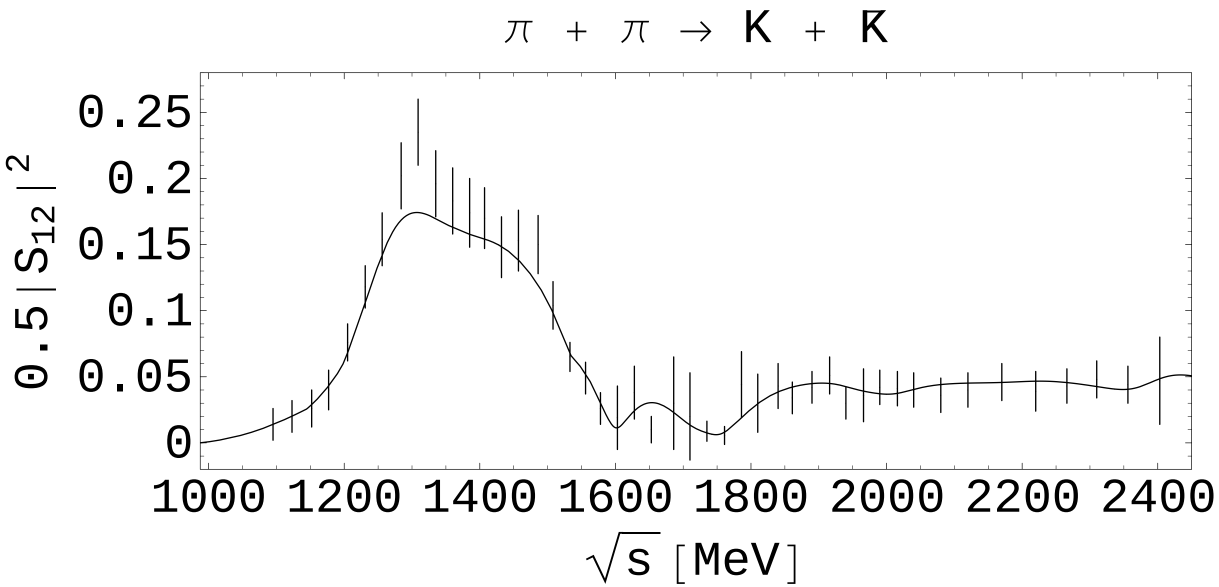
The data for $\pi\pi \to K\overline{K}, \eta\eta$ are taken from works (*S.J.Lindenbaum, R.S.Longacre, PL B* ***274*** *(1992) 492; R.S.Longacre* et al., *PL B* ***177*** *(1986) 223*).

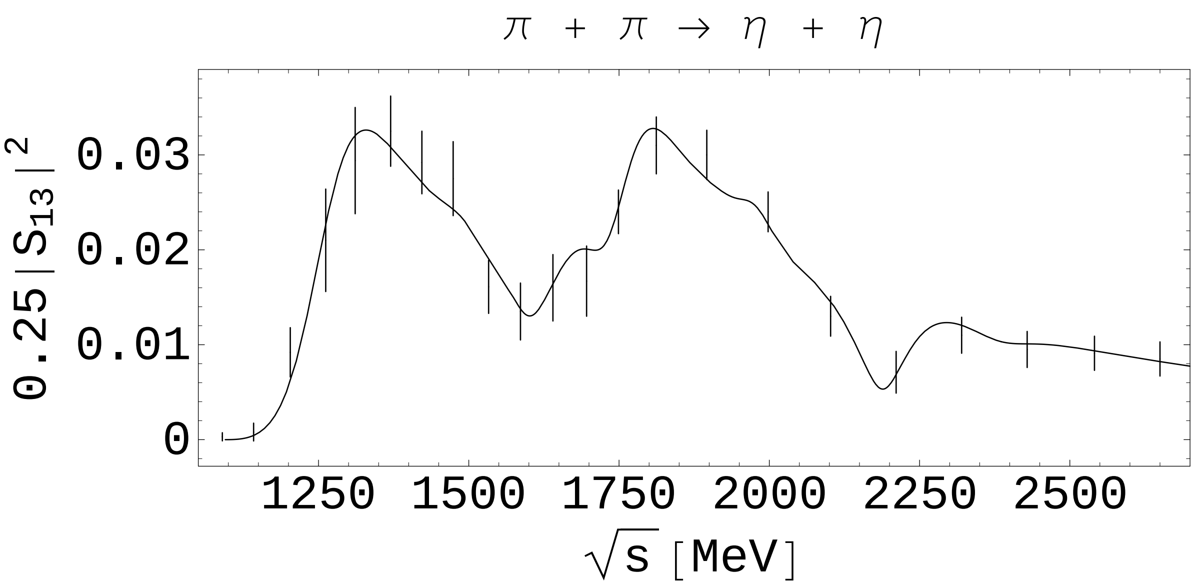
A satisfactory description (the total χ^2/NDF $= 161.147/(168-65) \approx 1.56$) is obtained already with ten resonances $f_2(1270)$, $f_2(1430)$, $f_2'(1525)$, $f_2(1580)$, $f_2(1730)$, $f_2(1810)$, $f_2(1960)$, $f_2(2000)$, $f_2(2240)$ and $f_2(2410)$ and also with eleven states when adding one more resonance $f_2(2020)$ which is needed in the combined analysis of processes $p\bar{p} \to \pi\pi, \eta\eta, \eta\eta'$ (*V.V.Anisovich et al., IJMP A **20 (2005) 6327*). Description in the latter case is practically the same one as in the case of ten resonances: the total χ^2/NDF $= 156.617/(168-69) \approx 1.58$.**

In the following figures we demonstrate our fitting to the data.



The phase shift and module of the $\pi\pi$-scattering D-wave matrix element.





The squared modules of the $\pi\pi \to K\overline{K}$ (upper figure) and $\pi\pi \to \eta\eta$ (lower figure) D-wave matrix elements.

The parameters of Breit–Wigner forms for 10 states (in MeV).

State	M_r	f_{r1}	f_{r2}	f_{r3}	f_{r4}
$f_2(1270)$	1275.3±1.8	470.8±5.4	201.5±11.4	90.4±4.76	22.4±4.6
$f_2(1430)$	1450.8±18.7	128.3±45.9	562.3±142	32.7±18.4	8.2±65
$f_2'(1525)$	1535±8.6	28.6±8.3	253.8±78	92.6±11.5	41.6±160
$f_2(1600)$	1601.4±27.5	75.5±19.4	315±48.6	388.9±27.7	127±199
$f_2(1710)$	1723.4±5.7	78.8±43	289.5±62.4	460.3±54.6	107.6±76.7
$f_2(1810)$	1761.8±15.3	129.5±14.4	259±30.7	469.7±22.5	90.3±90
$f_2(1960)$	1962.8±29.3	132.6±22.4	333±61.3	319±42.6	65.4±94
$f_2(2000)$	2017±21.6	143.5±23.3	614±92.6	58.8±24	450.4±221
$f_2(2240)$	2207±44.8	136.4±32.2	551±149	375±114	166.8±104
$f_2(2410)$	2429±31.6	177±47.2	411±196.9	4.5±70.8	460.8±209

The background are: $\alpha_{11} = -0.07805$, $\alpha_{12} = 0.03445$, $\alpha_{10} = -0.2295$, $\beta_1 = -0.0715$, $\gamma_1 = -0.04165$, $\beta_2 = -0.981$, $\gamma_2 = 0.736$, $\beta_3 = -0.5309$, $\gamma_3 = 0.8223$.

The parameters of Breit–Wigner forms for 11 states.

State	M_r	f_{r1}	f_{r2}	f_{r3}	f_{r4}
$f_2(1270)$	1276.3±1.8	468.9±5.5	201.6±11.6	89.9±4.79	7.2±4.6
$f_2(1430)$	1450.5±18.8	128.3±45.9	562.3±144	32.7±18.6	8.2±63
$f_2'(1525)$	1534.7±8.6	28.5±8.5	253.9±79	89.5±12.5	51.6±155
$f_2(1600)$	1601.5±27.9	75.5±19.6	315±50.6	388.9±28.6	127±190
$f_2(1710)$	1719.8±6.2	78.8±43	289.5±62.6	460.3±545.	108.6±76.
$f_2(1810)$	1760±17.6	129.5±14.8	259±32.	469.7±25.2	90.3±89.5
$f_2(1960)$	1962.2±29.8	132.6±23.3	331±61.5	319±42.8	62.4±91.3
$f_2(2000)$	2006±22.7	155.7±24.4	169.5±95.3	60.4±26.7	574.8±211
$f_2(2020)$	2027±25.6	50.4±24.8	441±196.7	58±50.8	128±190
$f_2(2240)$	2202±45.4	133.4±32.6	545±150.4	381±116	168.8±103
$f_2(2410)$	2387±33.3	175±48.3	395±197.7	24.5±68.5	462.8±211

The background parameters are: $\alpha_{11} = -0.0755$, $\alpha_{12} = 0.0225$, $\alpha_{10} = -0.2344$, $\beta_1 = -0.0782$, $\gamma_1 = -0.05215$, $\beta_2 = -0.985$, $\gamma_2 = 0.7494$, $\beta_3 = -0.5162$, $\gamma_3 = 0.786$.

The resonance poles on sheets II, IV, VIII, and XVI for eleven states.

$\sqrt{s_r} = \mathrm{E}_r - i\Gamma_r/2$ in MeV is given.

	II		IV		VIII		XVI	
State	E_r	$\Gamma_r/2$	E_r	$\Gamma_r/2$	E_r	$\Gamma_r/2$	E_r	$\Gamma_r/2$
$f_2(1270)$	1282±2.6	67.5±4.2	1257±3.5	99.6±3	1277±3	73.4±4	1264±3.4	98±3.5
$f_2(1430)$	1425±48	98.8±54	1421±49	109±53	1426±48	98±55	1422±49	109±52
$f_2'(1525)$	1534±13	24±28	1534±13	23±9	1534±13	17±29	1534±13	19.5±28
$f_2(1600)$	1590±44	80.5±34	1592±41	74±34	1600±41	23±35	1601±40	9.4±35
$f_2(1710)$	1710±12	87±27	1711±11	84±27	1717±9.6	42.4±27	1718±9	32±27
$f_2(1810)$	1752±26	79±15	1752±26	84±15	1757±25	50.6±15	1758±25	36.5±15
$f_2(1960)$	1958±43	50±19	1957±43	57±19	1962±42	3.5±19	1962±42	7.4±19
$f_2(2000)$	2003±36	84±62	2004±35	68±64	2003±35	82±64	2002±36	95±62
$f_2(2020)$	2025±39	52±51	2026±38	45.4±57	2026±38	42.5±57	2025±39	52±51
$f_2(2240)$	2196±62	103±54.5	2197±62	98±55	2202±61	24±57	2201±62	45±57
$f_2(2410)$	2385±49	71±58	2387±47	5.6±61	2387±48	18.7±60	2385±49	84±59

The masses m_{res} and total widths Γ_{tot} of states are calculated from the pole positions using the denominator of the resonance part of amplitude in the form

$$T^{res} = \sqrt{s}\Gamma_{el}/(m_{res}^2 - s - i\sqrt{s}\Gamma_{tot}).$$

$$m_{res} = \sqrt{\mathrm{E}_r^2 + (\Gamma_r/2)^2}, \quad \Gamma_{tot} = \Gamma_r.$$

The masses and total widths of the f_2-resonances (all in MeV).

	$f_2(1270)$	$f_2(1430)$	$f_2'(1525)$	$f_2(1600)$	$f_2(1710)$	$f_2(1810)$
m_{res}	1268.0±3.4	1425.5±49.2	1533.8±13.4	1592.3±44.3	1712.2±11.6	1753.8±25.6
Γ_{tot}	196.0±7.0	218.6±105.4	48.4±56.0	161.0±68.6	174.0±53.8	167.6±29.4
Sheet	XVI	IV, XVI	II, IV	II	II	IV

	$f_2(1960)$	$f_2(2000)$	$f_2(2020)$	$f_2(2240)$	$f_2(2410)$
m_{res}	1958.0±42.9	2004.0±36.3	2026.0±39.0	2198.8±62.3	2386.0±48.7
Γ_{tot}	113.6±37.0	189.2±123.2	104.4±102.2	205.6±109.0	167.6±117.0
Sheet	IV	XVI	II, XVI	II	XVI

It is clear that the values of these quantities, calculated from the pole positions on various sheets, slightly differ from each other; for the $f_2(2240)$ and $f_2(2410)$, lying in the energy region where data are very scanty, even considerably. There are shown only the values which match best the corresponding values M_r and the quantities $\sum_{i=1}^{N} f_{ri}^2/M_r$. The sheets on which the poles, used in calculation of m_{res} and Γ_{tot}, lie are also indicated. In those cases when two sheets are indicated, the pole positions on these sheets do not differ more than 1-1.5 MeV.

Analysis of the isovector F-wave of $\pi\pi$ scattering

When analyzing of the $\pi\pi$-scattering data in the $I^G J^{PC} = 1^+3^{--}$ sector (*B. Hyams et al., NP B **64** (1973) 134*) and taking into account that the most considerable modes of decay of the $\rho_3(1690)$ are $\pi\pi$, 4π, $\omega\pi$, $K\overline{K}$ and $K\overline{K}\pi$, we used 4-channel Breit–Wigner forms in constructing the Jost matrix determinant $d(\sqrt{s-s_1}, \sqrt{s-s_2}, \sqrt{s-s_3}, \sqrt{s-s_4})$ where s_1, ..., s_4 are respectively the thresholds above indicated up to the $K\overline{K}$. The resonance poles and zeros in the S-matrix are generated utilizing the Le Couteur–Newton relation

$$S_{11} = d(-\sqrt{s-s_1}, \cdots, \sqrt{s-s_4})/d(\sqrt{s-s_1}, \cdots, \sqrt{s-s_4}) \, .$$

$$d_{res}(s) = \prod_r \left[M_r^2 - s - i \sum_{j=1}^{4} \left(\sqrt{\frac{s-s_j}{M_r^2 - s_j}} \right)^7 R_{rj} \; f_{rj}^2 \right] .$$

The Blatt–Weisskopf factor for a particle with $J = 3$ is

$$R_{rj} = \frac{15 + 3(\sqrt{M_r^2 - s_j}\ r_{rj})^2 + \frac{2}{5}(\sqrt{M_r^2 - s_j}\ r_{rj})^4 + \frac{1}{15}(\sqrt{M_r^2 - s_j}\ r_{rj})^6}{15 + 3(\sqrt{s - s_j}\ r_{rj})^2 + \frac{2}{5}(\sqrt{s - s_j}\ r_{rj})^4 + \frac{1}{15}(\sqrt{s - s_j}\ r_{rj})^6}$$

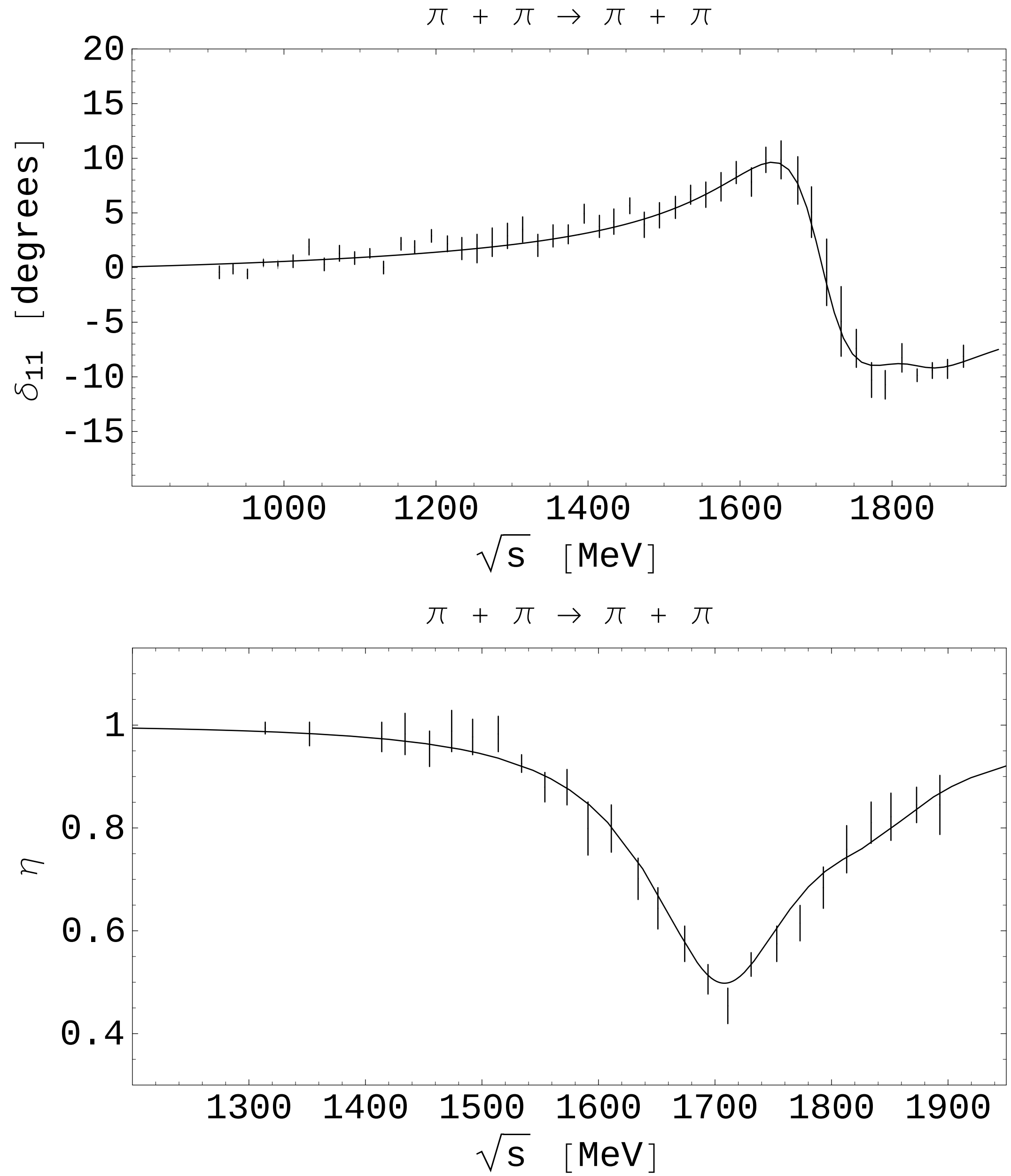
with radii of 0.927 fm in all channels.

The background part turned out to be elastic:

$$d_B = \exp\left[-i\left(\sqrt{\frac{s - 4m_\pi^2}{s}}\right)^7 a_1\right].$$

$a_1 = -0.0138 \pm 0.0011.$

The analysis is performed considering one and two resonances. The good description is obtained in both cases: the total $\chi^2/\mathrm{NDF} \approx 1$.



The phase shift and module of the $\pi\pi$-scattering F-wave matrix element.

The parameters of the Breit–Wigner forms for two ρ_3-like states (all in MeV).

State	M_r	f_{r1}	f_{r2}	f_{r3}	f_{r4}
$\rho_3(1690)$	1707.8±13.7	284.4±15.9	435.3±21.0	208.6±18.4	113.5±25
$\rho_3(1950)$	1833.5±28.6	96.3±18.3	331.8±28.0	297.7±16.5	110.4±28.3

The poles, generated by the Breit–Wigner forms on sheets II, IV, VIII, and XVI. $\sqrt{s_r} = E_r - i\Gamma_r/2$ in MeV is given.

	II		IV		VIII		XVI	
State	E_r	$\Gamma_r/2$	E_r	$\Gamma_r/2$	E_r	$\Gamma_r/2$	E_r	$\Gamma_r/2$
$\rho_3(1690)$	1705±5.6	48±8	1707.6±4.5	15.3±13	1703.6±3.9	70±14	1700.5±4.4	87.7±13.5
$\rho_3(1950)$	1830.4±28	55±14	1833.5±29	0.0±22.7	1833.5±27.5	11.7±15	1831±24.3	53.3±22.3

The parameters of the $\rho_3(1690)$ and its branching ratios compared with the averaged values from the PDG tables.

Scenario	m_{res}[MeV]	Γ_{tot}[MeV]	$\Gamma_{\pi\pi}/\Gamma_{tot}$	$\Gamma_{\pi\pi}/\Gamma_{4\pi}$	$\Gamma_{K\overline{K}}/\Gamma_{\pi\pi}$	$\Gamma_{\omega\pi}/\Gamma_{4\pi}$	$\Gamma_{K\overline{K}}/\Gamma_{tot}$
1 state	1703±4	179±12	0.29±0.022	0.472±0.097	0.146±0.06	0.235±0.04	0.042±0.03
2 states	1702.7±4	175±11	0.271±0.021	0.427±0.096	0.159±0.045	0.23±0.04	0.043±0.032
PDG	1688.8±2.1	160±10	0.243±0.013	0.332±0.026	$0.118^{+0.039}_{-0.032}$	0.23±0.05	0.013±0.0024

Discussion and conclusions

- In the $I^G J^{PC} = 0^+2^{++}$ sector, two analysis – without and with the $f_2(2020)$ – were carried out. We do not obtain $f_2(1640)$, $f_2(1910)$, $f_2(2150)$ and $f_2(2010)$, however, we see $f_2(1450)$ and $f_2(1710)$ which are related to the statistically-valued experimental points.

- Usually one assigns to the 1st tensor nonet the states $f_2(1270)$ and $f_2'(1525)$. To the 2nd nonet, one could assign $f_2(1600)$ and $f_2(1710)$ though for now the isodoublet member is not discovered. If $a_2(1730)$ is the isovector of this octet and if $f_2(1600)$ is almost its eighth component, then, from the GM-O formula
$$M_{K_2^*}^2 = \frac{1}{4}(3M_{f_2(1600)}^2 + M_{a_2(1730)}^2),$$
we would expect this isodoublet mass at about 1633 MeV. In the relation for masses of nonet
$$M_{f_2(1600)} + M_{f_2(1710)} = 2M_{K_2^*(1633)},$$
the left-hand side is only by 1.2% bigger than the right one.

In (*V.M.Karnaukhov et al., Yad.Fiz. **63, 652 (2000)*), one has observed in the mode $K_s^0\pi^+\pi^-$ the strange isodoublet with yet indefinite remaining quantum numbers and with mass 1629 ± 7 MeV. This state might be the tensor isodoublet of the second nonet.**

- **The states $f_2(1963)$ and $f_2(2207)$ together with the isodoublet $K_2^*(1980)$ could be put into the third nonet. Then in the relation for masses of nonet**

$$M_{f_2(1963)} + M_{f_2(2207)} = 2M_{K_2^*(1980)},$$

the left-hand side is only by 5.3% bigger than the right one.

If one consider $f_2(1963)$ as the eighth component of octet, the GM-O formula gives $M_{a_2} = 2030$ MeV. This value coincides with the one for a_2-meson obtained in analysis (*A.V.Anisovich et al., PL B **452, 173 (1999); ibid., **452**, 187 (1999); ibid., **517**, 261 (2001).*). This state is interpreted as a second radial excitation of the 1^-2^{++}-state on the basis of consideration of the a_2 trajectory on the (n, M^2) plane (*V.V.Anisovich et al.. IJMP A **20**, 6327 (2005)*).**

- **As to the $f_2(2000)$, the presence of the $f_2(2020)$ in the analysis with eleven resonances helps to interpret $f_2(2000)$ as the glueball. In the case of ten resonances, the ratio of the $\pi\pi$ and $\eta\eta$ widths is in the limits obtained in Ref.(*V.V.Anisovich et al., IJMP A* ***20****, 6327 (2005)*) for the tensor glueball on the basis of the $1/N_c$-expantion rules. However, the $K\overline{K}$ width is too large for the glueball. At practically the same description of processes with the consideration of eleven resonances as in the case of ten, their parameters have varied not much, except for the $f_2(2000)$ and $f_2(2410)$. Mass of the latter has decreased by about 40 MeV. As to the $f_2(2000)$, its $K\overline{K}$ width has changed significantly. Now all the obtained ratios of the partial widths are in the limits corresponding to the glueball.**

The question of interpretation of the $f_2(2020)$ and $f_2(2410)$ is open.

- **Finally we have $f_2(1450)$ and $f_2(1710)$ which are neither $q\bar{q}$ states nor glueballs. Since one predicts that masses of the lightest $q\bar{q}g$ hybrids are bigger than the ones of lightest glueballs, these states might be the 4-quark ones. Then for the isodoublet mass of the corresponding nonet, we would expect the value about 1570-1600 MeV. For now we do not know experimental indications for the tensor isodoublet of that mass. However, in the known experimental spectrum of the K_2^* family, there is a 500-MeV unoccupied gap from 1470 to 1970 MeV (*PDG'2008*), except for the above work (*V.M.Karnaukhov et al., Yad.Fiz. **63**, 652 (2000)*). Also, as one can see in the PDG tables on the $a_2(1700)$ listing, the observed isovector tensor states in the 1660-1775-MeV interval differ in the width by about 2-3 times, i.e., possess various properties. E.g., the broad state, observed in $\bar{p}p \to \eta\eta\pi^0$ (*I. Uman et al. (FNAL E835), PR D **73** (2006) 052009*) with mass 1702 ± 7 MeV and width 417 ± 19 MeV, might be the isovector member of the corresponding 4-quark nonet.**

Assumption of this possibility presupposes an existence of the scalar tetraquarks at lower energies (*R.L. Jaffe, PR D* ***15* *(1977) 267, 281; N.N. Achasov et al., PL B* ***96*** *(1980) 168; Z. Phys. C* ***22*** *(1984) 53*) which are not seen in our analysis (*SBKN-PRD'2010*). One can think that these states are a part of the background in view of their very large widths.**

- **The analysis of the F-wave $\pi\pi$ scattering data (*B. Hyams et al., NP B* ***64*** *(1973) 134*) indicates that, except the known $\rho_3(1690)$ (in our analysis $m_{res} \approx 1703$ MeV, $\Gamma_{tot} \approx 175$ MeV), there might be one more the state lying above 1830 MeV. Since the $\pi\pi$ scattering data above 1890 MeV are absent, it is impossible to say something serious on parameters of this state. However the $\rho_3(1950)$ is not contradict to the data and even improves a little the obtained parameters of the $\rho_3(1690)$ and its branching ratios when comparing them with the PDG tables (*C.Amsler et al. (PDG), PL B* ***667*** *(2008) 1*).**