ON NATURE OF SYMMETRIES

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Classical Hamiltonian mechanics

The Gibbs distribution

$$G = Z^{-1} e^{-\beta H(p,q)}, \quad \beta = \frac{1}{kT},$$

Variations of canonical variables

$$\delta x^i = \delta x^i_\perp + \delta x^i_{||}, \quad (x^i) = (p, q)$$

1. $H(x + \delta x_\perp) = H(x) \;\rightarrow\; \partial_i H(x) \delta x^i_\perp = 0 \;\rightarrow\; \delta x^i_\perp = \omega^{ij} \partial_j H \delta t\,, \;\; \omega^{ij} = -\omega^{ji}$

$$t - time \;\rightarrow\; \boxed{\dot{x}^i = \omega^{ij} \partial_j H}, \quad \dot{x} = \frac{dx}{dt}.$$ Time arrow.

2. $H(x + \delta x_{||}) \neq H(x) \rightarrow \;\; non-equilibrium \;\; distributions$

Harmonic oscillator (HO). Quantum mechanics (QM)

$$G_2 = \frac{1}{2\pi\hbar} e^{-\beta\omega\frac{p^2+q^2}{2}}\,, \quad \hbar = \frac{1}{\beta\omega}\,,$$

Measure

$$d\mu(q,p) = \frac{dq\wedge dp}{h} e^{-\beta\omega\frac{p^2+q^2}{2}}\,, \; h = 2\pi\hbar\,, \; \int dp dq G_2(p,q) = 1 \; (definition\ of\ h)$$

Stability index (SI)

In a thermal bath the most stable distributions survive.

Compare $G_1 = \frac{1}{\sqrt{2\pi}} e^{-\frac{q^2}{2}}$ *and* $G_2 = \frac{1}{2\pi} e^{-\frac{p^2+q^2}{2}}$.

Any $q \to q + \delta q$ changes G_1; variations conserving $p^2 + q^2$ conserve G_2

SI : $r = \frac{I}{N}$, I – number of invariant variables, N – total number of variables

For G_1: $I = 0$, $r_1 = 0$

For G_2: $I = 1$, $N = 2$, $r_2 = \frac{1}{2}$

} $r_2 > r_1$, so G_2 is more stable

Notice : $p^2 + q^2$ is proportional to the HO Hamiltonian, and G_2 is more stable.

Strings and fields are made of HO.

The larger is r the larger is the time of relaxation t_r.

Groups, representations

Groups :	$SU(l+1)$	$SO(2l+1)$	$SO(2l)$	$Sp(2l)$	G_2	F_4	E_6	E_7	E_8
Number of invariants :	l	l	l	l	2	4	6	7	8
Dimension of group N:	$l(l+2)$	$l(2l+1)$	$l(2l-1)$	$l(2l+1)$	14	52	78	133	248
SI: $r=\frac{l}{N}$	$\frac{1}{l+2}$	$\frac{1}{2l+1}$	$\frac{1}{2l-1}$	$\frac{1}{2l+1}$	$\frac{1}{7}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{19}$	$\frac{1}{31}$

Groups $\mathrm{SU}(\mathrm{n})$ are preferable

Simplest representations of groups $\mathrm{SU}(\mathrm{n})$

Representations:	Fundamental	Adjoint
Dimensions (N):	n	n^2-1
SI: $r=\frac{l}{N}=\frac{n-1}{N}$	$\frac{n-1}{n}$	$\frac{n-1}{n^2-1}=\frac{1}{n+1}$

$$\frac{n-1}{n} > \frac{1}{n+1},$$

for $n>1$

Conclusion: group: $\mathrm{SU}(\mathrm{n})$

Fundamental representation

Strings

Ordered sets of HO are more stable than disordered ones.
The Nambu – Goto string is most preferable because two degree of freedom of field (i.e. infinite number of degrees of freedom) disappears from Hamiltonian.

It introduces two new important notions

1. Pseudoeuclidean space

$$L = \sum_j \frac{1}{2}\left[\dot{q}_j^2 - \tilde{\gamma}(q_j - q_{j-1})^2\right] \to c\int dx \frac{1}{2}\left(\dot{\varphi}^2(t,x) - \varphi'^2(t,x)\right) \Rightarrow (\partial_t^2 - \partial_x^2)\varphi(t,x) = 0,$$

$$\bullet \; \underbrace{\bullet \; \bullet}_{a} \; \bullet \; \bullet \; \bullet \qquad ja \xrightarrow[j\to\infty]{} x, \quad \varphi' = \partial_x\varphi\,, \quad \frac{q}{\sqrt{a}} \xrightarrow[a\to 0]{} \varphi$$

2. Gauge transformations

$$S_{NG} = -\gamma\int d^2u\sqrt{-g} = -\gamma\int d^2u\left\{-\begin{vmatrix}\dot{X}\dot{X} & \dot{X}X' \\ X'\dot{X} & X'X'\end{vmatrix}\right\}^{1/2} = -\gamma\int d\tau d\sigma\,|\dot{t}|\left|\vec{k}\right|\sqrt{1-\vec{v}_\perp^2}$$

$$X\colon\; X^\mu(\tau,\sigma) \equiv \left(t(\tau,\sigma), \vec{r}(t(\tau,\sigma),\sigma)\right),\;\; \mu = 0,1,\ldots,25,\;\; \dot{X}\dot{X} \equiv \dot{X}^\mu\dot{X}^\mu,\; \dot{X} \equiv \frac{\partial X}{\partial\tau},\; X' \equiv \frac{\partial X}{\partial\sigma}$$

$$(u_1,u_2) = (\tau,\sigma),\;\; \dot{t} = \frac{\partial t(\tau,\sigma)}{\partial\tau},\;\; \vec{k} = \frac{\partial\vec{r}(t,\sigma)}{\partial\sigma},\;\; \vec{v}_\perp = \vec{v} - \vec{k}\frac{(\vec{v}\vec{k})}{\vec{k}^2},\;\; \vec{v} = \frac{\partial\vec{r}}{\partial t},\;\; \vec{k}\vec{v}_\perp = 0\,,$$

and one degree of freedom disappears. The action S_{NG} is invariant under gauge transformations.

Symmetries

1. SU(5) (GUT)

The Nambu – Goto string exists in space-time (25 + 1). The field X^{μ} has two non-physical degrees of freedom, so the number of physical degrees are 24. There are two groups of dimension 24: SO(24) and SU(5) ($24 = 5^2 - 1$), and

$$r_{SU(5)} = \frac{4}{24} = \frac{1}{6} > r_{SO(24)} = \frac{12}{276} = \frac{1}{23}$$

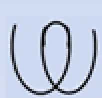
The fundamental representation of SU(5) is the most stable

2. SUSY

In 2D space string curves. $E^2 = \vec{P}^2$, $in\ QM$ $\vec{P}^2 \to -\Delta_2 = \hat{P}_r^2 + \frac{1}{r^2}\hat{P}_\varphi^2 - \frac{1}{4r^2}$,

$\hat{P}_r = \frac{1}{i}\frac{1}{\sqrt{r}}\partial_r\sqrt{r}$, $\hat{P}_\varphi = \frac{1}{i}\partial_\varphi$. For the string exitations $\hat{P}_r\psi_{ph} = 0$ (no radial motion)

$\hat{H}^2 = \hat{P}^2 - \frac{1}{4r^2} = \left(\hat{P} - \frac{1}{2r}\right)\left(\hat{P} + \frac{1}{2r}\right) \equiv \hat{H}_+\hat{H}_-$, $\hat{P} = \frac{1}{ir}\partial_\varphi$. If r = R – motion over the circle

$\hat{H}^2 \geq 0 \Rightarrow P_{min} = \frac{1}{2r}$.  ,  – helix

Degrees of freedom: ,  (independent) $\zeta = y_1 + iy_2 \to \zeta^{1/2}$ – spinor

$u_\pm = t \pm z \to u_\pm^{1/2} \equiv v_\pm$

3. Ramon – Neveu – Schwarz string

$$S_{RNS} = \int d\tau d\sigma \left(-\frac{\partial X^\mu}{\partial \sigma_+} \frac{\partial X^\mu}{\partial \sigma_-} + \bar{S}^\mu i \gamma_b \partial_b S^\mu \right), \quad \sigma_\pm = \tau \pm \sigma, \quad b = 0,1.$$

Supersymmetric string exists in 10 – dim. space – time

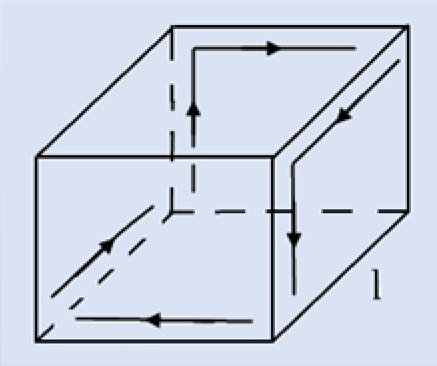
$10 - 2 = 8 = 3^2 - 1 \rightarrow \mathrm{SU}(3)$

Spectrum of $\hat{P} = \frac{\hbar}{ir}\frac{d}{d\varphi}$: $\hat{P}\psi_n = P_n \psi_n$, $\psi_n = re^{in\varphi}$, $P_n = \frac{\hbar}{r} n \rightarrow$ HO spectrum

$$\hat{P} = \frac{\hbar}{R} \hat{a}^+ \hat{a}, \quad \hat{a}^+ = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \sqrt{1} & 0 & 0 & \dots \\ 0 & \sqrt{2} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \hat{a} = \begin{pmatrix} 0 & \sqrt{1} & 0 & \dots \\ 0 & 0 & \sqrt{2} & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Kaluza – Klein – Mandel – Fock

The simple idea



$$g^{\mu\nu} = \eta^{\mu\nu} + \Delta^{\mu\nu}$$

$$\Delta^{\mu\nu}_{(\vec{n})} = \frac{1}{l^2}\frac{1}{L_c}\frac{1}{N_p}\sum_{\vec{n}}\oint d\sigma\, X^{\mu}(\tau,\sigma)X^{\nu}(\tau,\sigma)$$

$\vec{n}(n_1, \dots, n_p)$ – the center of the cube

L_c – the length of a contour

N_p – number of independent closed contours

Then

$$S = \int d^D x \sqrt{|g_D|} R^D$$

$$g_{\mu\nu} = \begin{pmatrix} g_{ij} + g_{ab}A_i^a A_j^b & A_i^a g_{ab} \\ A_j^b g_{ba} & g_{ab} \end{pmatrix}$$

$\mu,\nu = 0,1, \dots, D-1$

$i,j = 0,1, \dots, p$

$a,b = p+1, \dots, D-1$

Suppose: $\frac{\partial g_{\mu\nu}}{\partial x^a} = 0, \quad \frac{\partial g_{ab}}{\partial x^\mu} = 0.$

Then: $$S = V^{D-p-1}\int d^{p+1}x \sqrt{|g|}\left(R - \frac{1}{4}\hat{F}^2 + \cdots\right), \qquad g = \det g_{ij}$$

R – scalar curvature of $(p+1)D$ physical space – time

$$\hat{F}^2 = g_{ab}F^a_{ki}F^b_{lj}g^{kl}g^{ij}, \quad \hat{F}_{kj} = \left[D_k, D_j\right]$$

1. CLASSICAL MECHANICS.
    1) Hamiltonian mechanics.
    2) Time arrow.

2. QUANTUM MECHANICS.
    1) Probability amplitudes.
    2) The Planck constant h.
    3) The Schrödinger equation.
    4) The Fock space.
    5) CPT-invariance.

3. QUANTUM FIELD THEORY.
    1) Absence of divergences.
    2) Breaking of chiral semmetry.

4. STRINGS.

1) Anticommuting variables.
2) Fermions (spin $\hbar/2$).
3) SUSY.
4) Model of superstring.
5) Appearance of symmetries.
6) Groups $\mathrm{SU}(5)$ (broken), $\mathrm{SU}(3)$ (exact).
7) "The Dirac basement".
8) Gauge symmetries.
9) Pseudoeuclidean space-time.

5. COSMOLOGY.

1) The cosmological constant.
2) Dark matter.
3) Model of black hole.
4) Unification of interactions (Kaluza-Klein-Mandel-Fock).
5) Solution of the problem of compactification.
6) Inflation.
7) The problem of horizon.