GENERALIZED DYNAMICS OF RARITA–SCHWINGER FIELD

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Introduction

The Rarita–Schwinger field is used to describe particles with spin 3/2. It is well know that this field contains more degrees of freedom (DOF) than one need to describe that particles.
One of ways to eliminate extra DOF is to put some constrains on the field. But this approach leads to several difficulties when interaction is taken into account. The other way is to use Rarita–Schwinger field with extra DOF even with interactions are on and require that these extra DOF are to be unphysical only further in calculations.
In this work we consider the most general Lagrangian for Rarita–Schwinger field, derive wave equation for extra components of Rarita–Schwinger field, and calculate the projection operators for extra components of inverse propagator. Under «generalized dynamics» here we mean that all components of the field are taken into account.

Lagrangian

The free Lagrangian of the Rarita-Schwinger field is defined by differential operator $S^{\mu\nu}$, which is in fact the inverse propagator

$$\mathscr{L} = \overline{\Psi}^{\mu} S_{\mu\nu} \Psi^{\nu}. \tag{1}$$

The Lagrangian should fulfill the main requirements:

- The fermion Lagrangian is linear in derivatives.
- It should be hermitian $\mathscr{L}^{\dagger} = \mathscr{L}$ or $\gamma^0 (S^{\mu\nu})^{\dagger} \gamma^0 = S^{\nu\mu}$.
- The spin-3/2 contribution has standard pole form.
- Lagrangian should not be singular at $p \to 0$.

General Lagrangian of Rarita–Schwinger field

The Lagrangian written below satisfies all the necessary requirements.

$$S^{\mu\nu} = g^{\mu\nu}(\hat{p} - M) + p^{\mu}\gamma^{\nu}(r_5 + r_4 - 1 + \imath a_5) + \\ + p^{\nu}\gamma^{\mu}(r_5 + r_4 - 1 - \imath a_5) + \gamma^{\mu}\gamma^{\nu}(M + r_1) - \gamma^{\mu}\hat{p}\gamma^{\nu}(r_4 - 1). \quad (2)$$

Here M is mass of spin 3/2 particle, the r_1, r_4, r_5 and a_5 are arbitrary real parameters. These four are related only with extra DOF of the field.

Field transformation

There exists the well-known transformation of Rarita–Schwinger field

$$\Psi_\mu \to \Psi'_\mu : \ \Psi_\mu = \theta_{\mu\nu}(B)\Psi'^{\nu}, \tag{3}$$

where $\theta_{\mu\nu}(B) = g_{\mu\nu} + B\gamma_\mu\gamma_\nu$ and $B = b + \imath\beta$ is a complex parameter.
This transformation keeps all requirements that are imposed on Lagrangian, so when applied it to inverse propagator (2)

$$S_{\mu\nu} \to S'_{\mu\nu} = \theta_{\mu\alpha}(B^*)S^{\alpha\beta}\theta_{\beta\nu}(B), \tag{4}$$

this means that this transformation is in fact the reparametrization — after transformation of the (2) one obtain the same operator with changed parameters

$$\theta_{\mu\alpha}(B^*)S^{\alpha\beta}(r_1, r_4, r_5, a_5)\theta_{\beta\nu}(B) = S_{\mu\nu}(r'_1, r'_4, r'_5, a'_5). \tag{5}$$

The Λ-basis

To deal with spin-tensors like inverse propagator $S_{\mu\nu}$ in convenient and systematic way is to use the so-called Λ-basis. The Λ-basis are defined by its elements which we write in following form

$$\begin{aligned}
\mathcal{P}_{1\mu\nu} &= \Lambda^+ \mathcal{P}^{3/2}_{\mu\nu}, & \mathcal{P}_{2\mu\nu} &= \Lambda^- \mathcal{P}^{3/2}_{\mu\nu}, \\
\mathcal{P}_{3\mu\nu} &= \Lambda^+ \mathcal{P}^{1/2}_{11\mu\nu}, & \mathcal{P}_{4\mu\nu} &= \Lambda^- \mathcal{P}^{1/2}_{11\mu\nu}, \\
\mathcal{P}_{5\mu\nu} &= \Lambda^+ \mathcal{P}^{1/2}_{22\mu\nu}, & \mathcal{P}_{6\mu\nu} &= \Lambda^- \mathcal{P}^{1/2}_{22\mu\nu}, \\
\mathcal{P}_{7\mu\nu} &= \Lambda^+ \mathcal{P}^{1/2}_{21\mu\nu}, & \mathcal{P}_{8\mu\nu} &= \Lambda^- \mathcal{P}^{1/2}_{21\mu\nu}, \\
\mathcal{P}_{9\mu\nu} &= \Lambda^+ \mathcal{P}^{1/2}_{12\mu\nu}, & \mathcal{P}_{10\mu\nu} &= \Lambda^- \mathcal{P}^{1/2}_{12\mu\nu},
\end{aligned} \tag{6}$$

The Λ-basis

where

$$
\begin{aligned}
(\mathcal{P}^{3/2})^{\mu\nu} &= g^{\mu\nu} - n_1^\mu n_1^\nu - n_2^\mu n_2^\nu \\
(\mathcal{P}^{1/2}_{11})^{\mu\nu} &= n_1^\mu n_1^\nu, \qquad (\mathcal{P}^{1/2}_{22})^{\mu\nu} = n_2^\mu n_2^\nu, \\
(\mathcal{P}^{1/2}_{21})^{\mu\nu} &= n_1^\mu n_2^\nu, \qquad (\mathcal{P}^{1/2}_{12})^{\mu\nu} = n_2^\mu n_1^\nu,
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
n_{1\mu} &= \frac{1}{\sqrt{3}}\left(\gamma_\mu - \frac{p_\mu \hat{p}}{p^2}\right) = \frac{1}{\sqrt{3}}\left(g_{\mu\lambda} - \frac{p_\mu p_\lambda}{p^2}\right)\gamma^\lambda, \\
n_{2\mu} &= p_\mu / \sqrt{p^2}\, I_4, \quad (n_i \cdot n_j) = \delta_{ij} I_4,
\end{aligned} \tag{8}
$$

$$
\Lambda^\pm = \frac{\sqrt{p^2} \pm \hat{p}}{2\sqrt{p^2}}. \tag{9}
$$

Multiplication table

The advantage of this basis could be seen from its multiplication table:

	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$	$\mathcal{P}_5$	$\mathcal{P}_6$	$\mathcal{P}_7$	$\mathcal{P}_8$	$\mathcal{P}_9$	$\mathcal{P}_{10}$
$\mathcal{P}_1$	$\mathcal{P}_1$	0	0	0	0	0	0	0	0	0
$\mathcal{P}_2$	0	$\mathcal{P}_2$	0	0	0	0	0	0	0	0
$\mathcal{P}_3$	0	0	$\mathcal{P}_3$	0	0	0	$\mathcal{P}_7$	0	0	0
$\mathcal{P}_4$	0	0	0	$\mathcal{P}_4$	0	0	0	$\mathcal{P}_8$	0	0
$\mathcal{P}_5$	0	0	0	0	$\mathcal{P}_5$	0	0	0	$\mathcal{P}_9$	0
$\mathcal{P}_6$	0	0	0	0	0	$\mathcal{P}_6$	0	0	0	$\mathcal{P}_{10}$
$\mathcal{P}_7$	0	0	0	0	0	$\mathcal{P}_7$	0	0	0	$\mathcal{P}_3$
$\mathcal{P}_8$	0	0	0	0	$\mathcal{P}_8$	0	0	0	$\mathcal{P}_4$	0
$\mathcal{P}_9$	0	0	0	$\mathcal{P}_9$	0	0	0	$\mathcal{P}_5$	0	0
$\mathcal{P}_{10}$	0	0	$\mathcal{P}_{10}$	0	0	0	$\mathcal{P}_6$	0	0	0

The inverse propagator in Λ-basis

The inverse propagator of most general Lagrangian has the following decomposition in the Λ-basis

$$S_{\mu\nu} = \sum_{i=1}^{10} \bar{S}_i \mathcal{P}^i_{\mu\nu},$$

where $\bar{S}_i$ are

$$\begin{aligned}
\bar{S}_1 &= -M + E, & \bar{S}_2 &= -M - E,\\
\bar{S}_3 &= (2M + 3r_1) + E(3r_4 - 2), & \bar{S}_4 &= (2M + 3r_1) - E(3r_4 - 2),\\
\bar{S}_5 &= r_1 + E(2r_5 + r_4), & \bar{S}_6 &= r_1 - E(2r_5 + r_4),\\
\bar{S}_7 &= \sqrt{3}\big[-(M + r_1) + E(r_5 - \imath a_5)\big], &&\\
\bar{S}_8 &= \sqrt{3}\big[(M + r_1) + E(r_5 - \imath a_5)\big], &&\\
\bar{S}_9 &= \sqrt{3}\big[(M + r_1) + E(r_5 + \imath a_5)\big], &&\\
\bar{S}_{10} &= \sqrt{3}\big[-(M + r_1) + E(r_5 + \imath a_5)\big]. &&
\end{aligned}$$

Derivation of the wave equation

Consider the action for Rarita–Schwinger field

$$\mathscr{A} = \int \mathscr{L} \mathrm{d}^4 x, \quad \mathscr{L} = \bar{\Psi}^{\mu} S_{\mu\nu} \Psi^{\nu}. \tag{10}$$

To obtain the wave equation for extra DOF we decompose the wave function Ψ_μ in this way

$$\Psi_\mu = n_{1\mu}\Psi_1 + n_{2\mu}\Psi_2 + \chi_\mu, \tag{11}$$

where «vectors» $n_{i\mu}$ are given by (8) and

$$\chi_\mu : \ n_{1\mu}\chi^\mu = 0, \ n_{2\mu}\chi^\mu = 0. \tag{12}$$

Wave equation

The variation of wave function Ψ_μ: $\Psi_\mu \to \Psi'_\mu = \Psi_\mu + \delta\Psi_\mu$ is expressed by components as follow

$$\Psi_1 \to \Psi_1 + \delta\Psi_1,\ \Psi_2 \to \Psi_2 + \delta\Psi_2,\ \chi_\mu \to \chi_\mu + \delta\chi_\mu.$$

From requirement that $\delta\mathscr{A} = 0$ we get wave equations for Ψ_i, $i = 1, 2$:

$$\begin{cases} \left(\bar{S}_3\Lambda^- + \bar{S}_4\Lambda^+\right)\Psi_1 + \left(\bar{S}_7\Lambda^- + \bar{S}_8\Lambda^+\right)\Psi_2 = 0, \\ \left(\bar{S}_9\Lambda^+ + \bar{S}_{10}\Lambda^-\right)\Psi_1 + \left(\bar{S}_5\Lambda^+ + \bar{S}_6\Lambda^-\right)\Psi_2 = 0. \end{cases} \tag{13}$$

Matrix form

One could write down the system (13) in matrix form:

$$\left[\Lambda^+ \begin{pmatrix} \bar{S}_4 & \bar{S}_8 \\ \bar{S}_9 & \bar{S}_5 \end{pmatrix} + \Lambda^- \begin{pmatrix} \bar{S}_3 & \bar{S}_7 \\ \bar{S}_{10} & \bar{S}_6 \end{pmatrix}\right] \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 0. \qquad (14)$$

Denoting

$$\mathbf{M} = -\sqrt{p^2} \begin{pmatrix} \bar{S}_4 - \bar{S}_3 & \bar{S}_8 - \bar{S}_7 \\ \bar{S}_9 - \bar{S}_{10} & \bar{S}_5 - \bar{S}_6 \end{pmatrix}^{-1} \begin{pmatrix} \bar{S}_3 + \bar{S}_4 & \bar{S}_7 + \bar{S}_8 \\ \bar{S}_9 + \bar{S}_{10} & \bar{S}_5 + \bar{S}_6 \end{pmatrix} \qquad (15)$$

one could rewrite the system in form

$$\left[\hat{p}\mathbf{E} - \mathbf{M}\right] \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 0, \qquad (16)$$

where **E** is unit matrix.

Construction of projection operators

The projection operators we would like to introduce are defined by extra DOF in inverse propagator and its eigenstates and eigenvalues. We need to find out such operators that

$$S_{\mu\alpha}\Gamma^{\alpha}_{\nu i} = \lambda_i \Gamma_{\mu\nu i},$$

where $\Gamma_{\mu\nu i}$ are projection operators, λ_i are eigenvalues.
To find the operators we will use the Λ-basis, so the decomposition of the inverse propagator $S_{\mu\nu}$ has form

$$S_{\mu\nu} = \sum_{i=1}^{10} \bar{S}_i \mathcal{P}^i_{\mu\nu}.$$

The first two basis elements $\mathcal{P}_1$, $\mathcal{P}_2$ corresponds to spin 3/2 contribution.

The rest eight elements that are group into to two groups: $\mathcal{P}_3$, $\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_{10}$ and $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_8$, $\mathcal{P}_9$ represent the extra DOF (spin 1/2 contributions).

These two group are similar in sense that they have the same algebraic structure.

	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$	$\mathcal{P}_5$	$\mathcal{P}_6$	$\mathcal{P}_7$	$\mathcal{P}_8$	$\mathcal{P}_9$	$\mathcal{P}_{10}$
$\mathcal{P}_1$	$\mathcal{P}_1$	0	0	0	0	0	0	0	0	0
$\mathcal{P}_2$	0	$\mathcal{P}_2$	0	0	0	0	0	0	0	0
$\mathcal{P}_3$	0	0	$\mathcal{P}_3$	0	0	0	$\mathcal{P}_7$	0	0	0
$\mathcal{P}_4$	0	0	0	$\mathcal{P}_4$	0	0	0	$\mathcal{P}_8$	0	0
$\mathcal{P}_5$	0	0	0	0	$\mathcal{P}_5$	0	0	0	$\mathcal{P}_9$	0
$\mathcal{P}_6$	0	0	0	0	0	$\mathcal{P}_6$	0	0	0	$\mathcal{P}_{10}$
$\mathcal{P}_7$	0	0	0	0	0	$\mathcal{P}_7$	0	0	0	$\mathcal{P}_3$
$\mathcal{P}_8$	0	0	0	0	$\mathcal{P}_8$	0	0	0	$\mathcal{P}_4$	0
$\mathcal{P}_9$	0	0	0	$\mathcal{P}_9$	0	0	0	$\mathcal{P}_5$	0	0
$\mathcal{P}_{10}$	0	0	$\mathcal{P}_{10}$	0	0	0	$\mathcal{P}_6$	0	0	0

Simple problem

Consider the problem: calculate projection operators for system which inverse propagator has decomposition

$$S = \sum_{i=1}^{4} \bar{S}_i \mathcal{P}^i, \tag{17}$$

where operators $\mathcal{P}_i$ have the following multiplication table

	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$
$\mathcal{P}_1$	$\mathcal{P}_1$	0	$\mathcal{P}_3$	0
$\mathcal{P}_2$	0	$\mathcal{P}_2$	0	$\mathcal{P}_4$
$\mathcal{P}_3$	0	$\mathcal{P}_3$	0	$\mathcal{P}_1$
$\mathcal{P}_4$	$\mathcal{P}_4$	0	$\mathcal{P}_2$	0

There is implementation for that operators, namely

$$\mathcal{P}_1 = \Lambda^+,\ \mathcal{P}_2 = \Lambda^-,\ \mathcal{P}_3 = \Lambda^+\gamma^5,\ \mathcal{P}_4 = \Lambda^-\gamma^5, \quad \Lambda^\pm = \frac{\sqrt{p^2} \pm \hat{p}}{2\sqrt{p^2}}.$$

Eigenvalues

To find out the projectors we will use the basis presented above. First, we decompose the Γ_i into this basis

$$\Gamma = \sum_{i=1}^{4} \alpha_i \mathcal{P}_i$$

and substitute this into equation

$$S\Gamma = \lambda\Gamma,$$

where S is of form (17).
Simplifying one obtain the system

$$\begin{cases} \bar{S}_1\alpha_1 + \bar{S}_3\alpha_4 = \lambda\alpha_1, \\ \bar{S}_2\alpha_2 + \bar{S}_4\alpha_3 = \lambda\alpha_2, \\ \bar{S}_1\alpha_3 + \bar{S}_3\alpha_2 = \lambda\alpha_3, \\ \bar{S}_2\alpha_4 + \bar{S}_4\alpha_1 = \lambda\alpha_4, \end{cases} \tag{18}$$

Eigenvalues

The eigenvalues λ are solution of the following equation

$$\det(\mathbf{S}-\lambda\mathbf{E})=0, \quad \mathbf{E}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \mathbf{S}=\begin{pmatrix}\bar{S}_1 & \bar{S}_3\\ \bar{S}_4 & \bar{S}_2\end{pmatrix}. \tag{19}$$

This is equation of second order so there are two eigenvalues and two projection operators. Of course, if the $\bar{S}_i$ depend on $W=\sqrt{p^2}$ the λ would depend too.

Projectors

From the system (18) one could express, for example, the α_3 and α_4 and construct projection operators Γ_i in given form

$$\Gamma_i = \alpha_1\Big(\mathcal{P}_1 - \frac{\bar{S}_4}{\bar{S}_2 - \lambda_i}\mathcal{P}_4\Big) + \alpha_2\Big(\mathcal{P}_2 - \frac{\bar{S}_3}{\bar{S}_1 - \lambda_i}\mathcal{P}_3\Big).$$

The requirement that these operators are projectors, i.e. $\Gamma_i\Gamma_j = \delta_{ij}\Gamma_j$ allows to determine the α_1 and α_2. Using the characteristic equation (19), substituting and simplifying one get the following expression for two projection operators

$$\Gamma_1 = \frac{1}{\lambda_2 - \lambda_1}\Big((\bar{S}_2 - \lambda_1)\mathcal{P}_1 - \bar{S}_4\mathcal{P}_4\Big) - \frac{1}{\lambda_2 - \lambda_1}\Big((\bar{S}_1 - \lambda_1)\mathcal{P}_3 - \bar{S}_4\mathcal{P}_2\Big),$$
$$\Gamma_2 = \frac{-1}{\lambda_2 - \lambda_1}\Big((\bar{S}_2 - \lambda_2)\mathcal{P}_1 - \bar{S}_4\mathcal{P}_4\Big) + \frac{1}{\lambda_2 - \lambda_1}\Big((\bar{S}_1 - \lambda_2)\mathcal{P}_3 - \bar{S}_4\mathcal{P}_2\Big).$$

Rarita–Schwinger field case

The considered simple example allows easily get equations for eigenvalues of extra DOF (spin 1/2 contributions) in inverse propagator for Rarita–Schwinger field, so

$$\det(\mathbf{S}_1 - \lambda \mathbf{E}) = 0, \quad \mathbf{S}_1 = \begin{pmatrix} \bar{S}_3 & \bar{S}_7 \\ \bar{S}_{10} & \bar{S}_6 \end{pmatrix},$$

$$\det(\mathbf{S}_2 - \eta \mathbf{E}) = 0, \quad \mathbf{S}_2 = \begin{pmatrix} \bar{S}_4 & \bar{S}_8 \\ \bar{S}_9 & \bar{S}_5 \end{pmatrix}.$$

In case of Rarita–Schwinger field we have four eigenvalues and four projectors that are grouped into two groups.

Projectors for spin 1/2 contributions

Now it is easy to write down the expression for projection operators, namely
for group $\mathcal{P}_3$, $\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_{10}$

$$\Gamma_1 = \frac{1}{\lambda_2 - \lambda_1}\Big((\bar{S}_6 - \lambda_1)\mathcal{P}_3 - \bar{S}_{10}\mathcal{P}_{10}\Big) + \frac{-1}{\lambda_2 - \lambda_1}\Big((\bar{S}_3 - \lambda_1)\mathcal{P}_6 - \bar{S}_7\mathcal{P}_7\Big)$$

$$\Gamma_2 = \frac{-1}{\lambda_2 - \lambda_1}\Big((\bar{S}_6 - \lambda_2)\mathcal{P}_3 - \bar{S}_{10}\mathcal{P}_{10}\Big) + \frac{1}{\lambda_2 - \lambda_1}\Big((\bar{S}_3 - \lambda_2)\mathcal{P}_6 - \bar{S}_7\mathcal{P}_7\Big)$$

for group $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_8$, $\mathcal{P}_9$

$$\Gamma_3 = \frac{1}{\eta_2 - \eta_1}\Big((\bar{S}_5 - \eta_1)\mathcal{P}_4 - \bar{S}_9\mathcal{P}_9\Big) + \frac{-1}{\eta_2 - \eta_1}\Big((\bar{S}_4 - \eta_1)\mathcal{P}_5 - \bar{S}_8\mathcal{P}_8\Big),$$

$$\Gamma_4 = \frac{-1}{\eta_2 - \eta_1}\Big((\bar{S}_5 - \eta_2)\mathcal{P}_4 - \bar{S}_9\mathcal{P}_9\Big) + \frac{1}{\eta_2 - \eta_1}\Big((\bar{S}_4 - \eta_2)\mathcal{P}_5 - \bar{S}_8\mathcal{P}_8\Big),$$

Conclusion

In this work we derive the wave equations for extra degrees of freedom of Rarita–Schwinger field. The next step will be to consider they contributions to observables and search conditions to make them unphysical.
Also we obtain projection operators for extra degrees of freedom (spin 1/2 contributions) in Rarita–Schwinger field. For Rarita–Schwigner field one have four eigenvalues and four projectors.
The Λ-basis used in this work helps to get results in convenient and systematic way.

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