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Solving Bethe-Salpeter Equation in Minkowski Space

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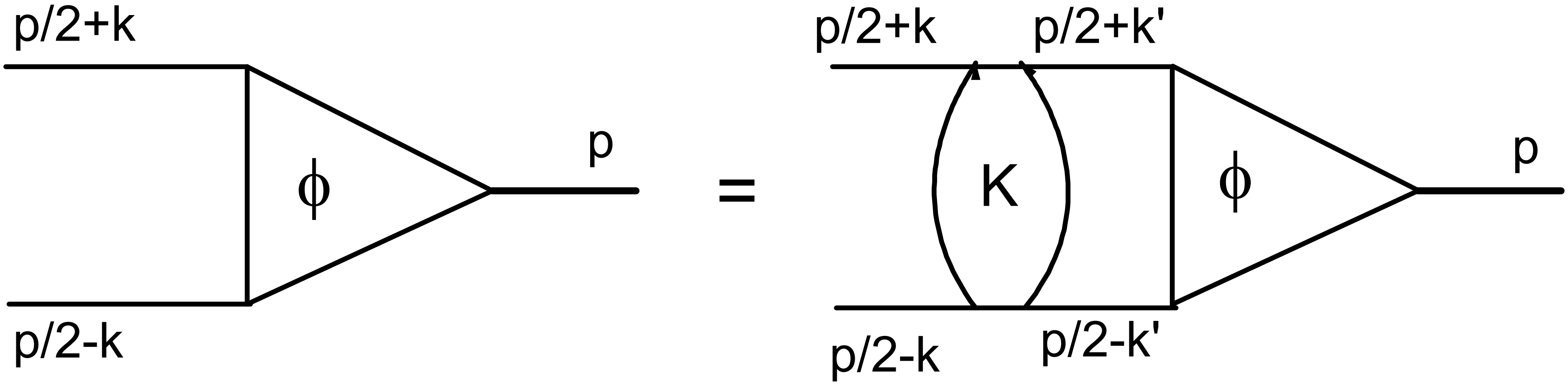
• Plan

- Solution for spinless particles
- Two fermions
- Application to electromagnetic form factors

• BS equation (1951)



$$\left((\frac{p}{2}+k)^2-m^2\right)\left((\frac{p}{2}-k)^2-m^2\right)\Phi(k,p)$$

$$= -i\int\frac{d^4k'}{(2\pi)^4}K(k,k',p)\Phi(k',p)$$

• Singularity

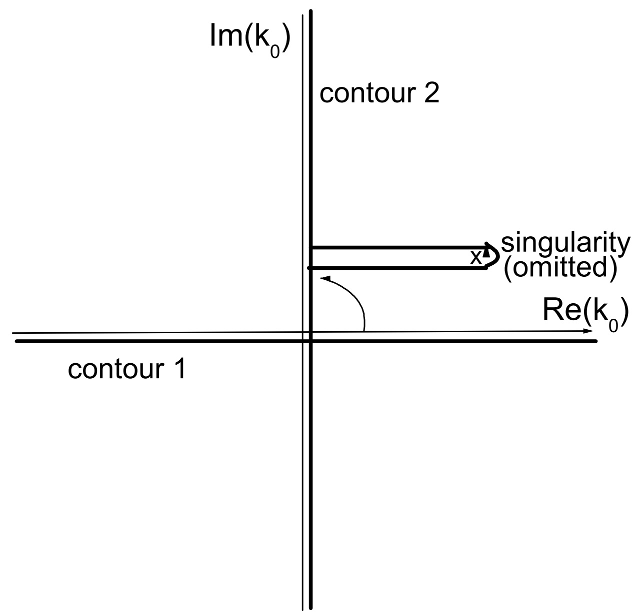
$$\Phi(k,p) = \frac{\Gamma(p,k)}{\left((\frac{p}{2}+k)^2 - m^2 + i\epsilon\right)\left((\frac{p}{2}-k)^2 - m^2 + i\epsilon\right)}$$

$$\Gamma(k,p) = -i\int \frac{d^4k'}{(2\pi)^4}\frac{K(k,k',p)\Gamma(p,k')}{\left((\frac{p}{2}+k')^2 - m^2 + i\epsilon\right)\left((\frac{p}{2}-k')^2 - m^2 + i\epsilon\right)}$$

It is not a problem in principle (it is normal).

But it is a problem for numerical solution.

• Wick rotation



$$\int \ldots d^4k = \int d^3k \int_{-\infty}^{\infty} \ldots dk_0 = \int_{-i\infty}^{i\infty} \ldots dk_0 = \int_{-\infty}^{\infty} \ldots i dk_4$$

• Euclidean space

Euclidean BS amplitude:

$$\Phi(\vec{k},k_0) \to \Phi_E(\vec{k},k_4) = \Phi(\vec{k},ik_0)$$

Euclidean BS equation (non-singular):

$$\left[\left(m^2 - \frac{M^2}{4} + \vec{k}^2 + k_4^2\right)^2 + M^2 k_4^2\right] \Phi_E(\vec{k},k_4)$$
$$= \int \frac{d^3k' dk_4}{(2\pi)^4} K_E(k,k')\Phi_E(\vec{k'},k_4)$$

It gives the bound state mass M.

But we need not only M , but also the BS amplitude in Minkowski space for practical applications (e.g., for em form factors).

• Aim

Our aim is to find not only the binding energies,
but the BS amplitude in Minkowski space.

• Separable kernel

V. Burov, S. Bondarenko, E. Rogochaya

Represent the kernel K in a separable form:

$$K(k,k',p) = \sum_{i=1}^{N} \tau_i(s) g_i(k,p) g_i(k',p)$$

BS equation is reduced to a system of linear (non-integral!) equations.
Solve it analytically, find BS amplitude in Minkowski space.

Applications to the np system
(deuteron, its electrodisintegration).

• Our (exact) method

V. A. Karmanov and J. Carbonell, Eur. Phys. J. **A27** (2006) 1.

K. Kusaka, A.G. Williams, (1995): spinless particles, ladder kernel only.

- Take BS amplitude in the Nakanishi form:

$$\Phi(k,p) = \int_{-1}^{1} dz' \int_{0}^{\infty} d\gamma' \frac{g(\gamma',z')}{\left[\gamma' + m^2 - \frac{1}{4}M^2 - k^2 - p{\cdot}k\; z' - i\epsilon\right]^3}$$

• Example

We set $g(\gamma,z) = 1$, calculate the integral and find

$$\Phi_M(k;p) = \frac{i^2}{\left[(\frac{p}{2}+k)^2 - m^2 + i\epsilon\right]\left[(\frac{p}{2}-k)^2 - m^2 + i\epsilon\right]},$$

i.e. just the product of two free propagators.

BS amplitude is, of course, still singular.

All the non-trivial dynamics is in the function $g(\gamma,z)$.

- Substitute it in the BS equation.

$$\Phi(k,p) = \frac{-i}{\left((\frac{p}{2}+k)^2 - m^2\right)\left((\frac{p}{2}-k)^2 - m^2\right)} \int \frac{d^4k'}{(2\pi)^4} K(k,k',p)\Phi(k',p)$$

- Apply to both sides of BS equation the LF projection

$$\int dk_- \Phi(k,p) = \int dk_- \frac{-i}{\left((\frac{p}{2}+k)^2 - m^2\right)\left((\frac{p}{2}-k)^2 - m^2\right)} \int \frac{d^4k'}{(2\pi)^4} K(k,k',p)\Phi(k',p)$$

  where $k_- = k_0 - k_z$

- Obtain a non-singular equation for $g(\gamma, z)$.

• Equation for $g(\gamma, z)$

(Obtained analytically, without any approximation.)

$$\int_0^\infty \frac{g(\gamma', z)d\gamma'}{\left[\gamma' + \gamma + z^2m^2 + (1 - z^2)\kappa^2\right]^2}$$

$$= \int_0^\infty d\gamma' \int_{-1}^1 dz' \, V(\gamma, z; \gamma', z')g(\gamma', z')$$

where $\kappa^2 = m^2 - \frac{1}{4}M^2$.

• This equation is equivalent to the initial BS equation.

Matrix form:

$$\lambda Bx = Ax$$

It is just standard form for well known fortran subroutine.

• Kernel

Given $K(k,k',p) \Rightarrow$ find $V(\gamma,z;\gamma',z')$:

$$
\begin{aligned}
V(\gamma,z;\gamma',z') &= \frac{p_+}{\pi}\int \frac{d^4k'}{(2\pi)^4}\frac{K(k,k',p)}{\left[k'^2+p\cdot k'z'-\gamma'-\kappa^2+i\epsilon\right]^3} \\
&\times \frac{dk_-}{\left[(\frac{p}{2}+k)^2-m^2+i\epsilon\right]\left[(\frac{p}{2}-k)^2-m^2+i\epsilon\right]},
\end{aligned}
$$

For any given BS kernel K we can calculate the kernel V of equation for $g(\gamma,z)$.

The method is valid for any kernel given by Feynman graphs.

• LF wave function

(As a by-product)

$$\psi(\vec{k}_\perp, x) = \int_{-\infty}^{\infty} \Phi(k,p) dk_- = \int_0^{\infty} \frac{g(\gamma', 1-2x) d\gamma'}{\left[\gamma' + \vec{k}_\perp^2 + m^2 - x(1-x)M^2\right]^2}$$

• OBE (ladder) kernel ($\mu \neq 0$)

One-boson exchange (ladder) kernel $K(k, k', p)$:

$$K(k, k', p) = \frac{-g^2}{(k - k')^2 - \mu^2 + i\epsilon}$$

Kernel $V(\gamma, z; \gamma', z')$:

$$V(\gamma, z; \gamma', z') = \frac{\alpha m^2 (1 - z)^2}{2\pi \left[\gamma + z^2 m^2 + (1 - z^2)\kappa^2\right]} \int_0^1 \frac{v^2 dv}{B_1^2}$$

$$\alpha = g^2/(16\pi m^2)$$

$B_1 = B_1(\gamma, z; \gamma', z'; v)$ is a polynomial. Integral $\int_0^1 \frac{v^2 dv}{B_1^2}$ is calculated analytically. Equation is solved numerically.

• Kernel $V(\gamma, z; \gamma', z')$ v.s. z'

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(* Spinless case  *)
z = 0.8
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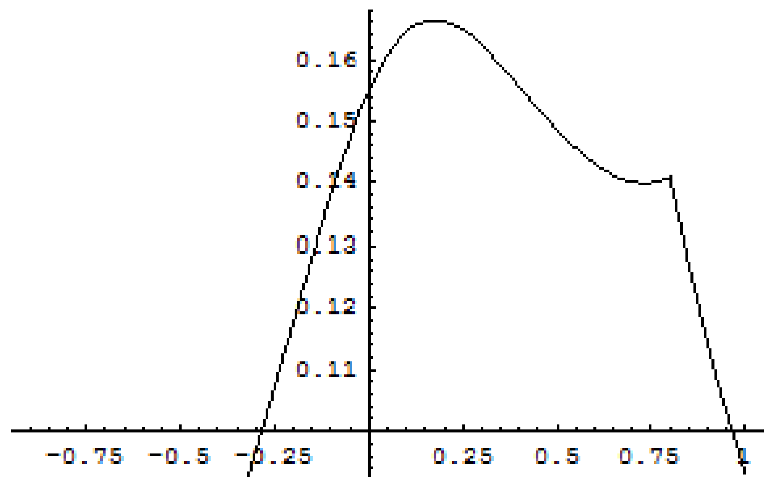


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- Graphics -
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Kernel $V(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.8$

• Numerical results (ladder, $\mu \neq 0$)

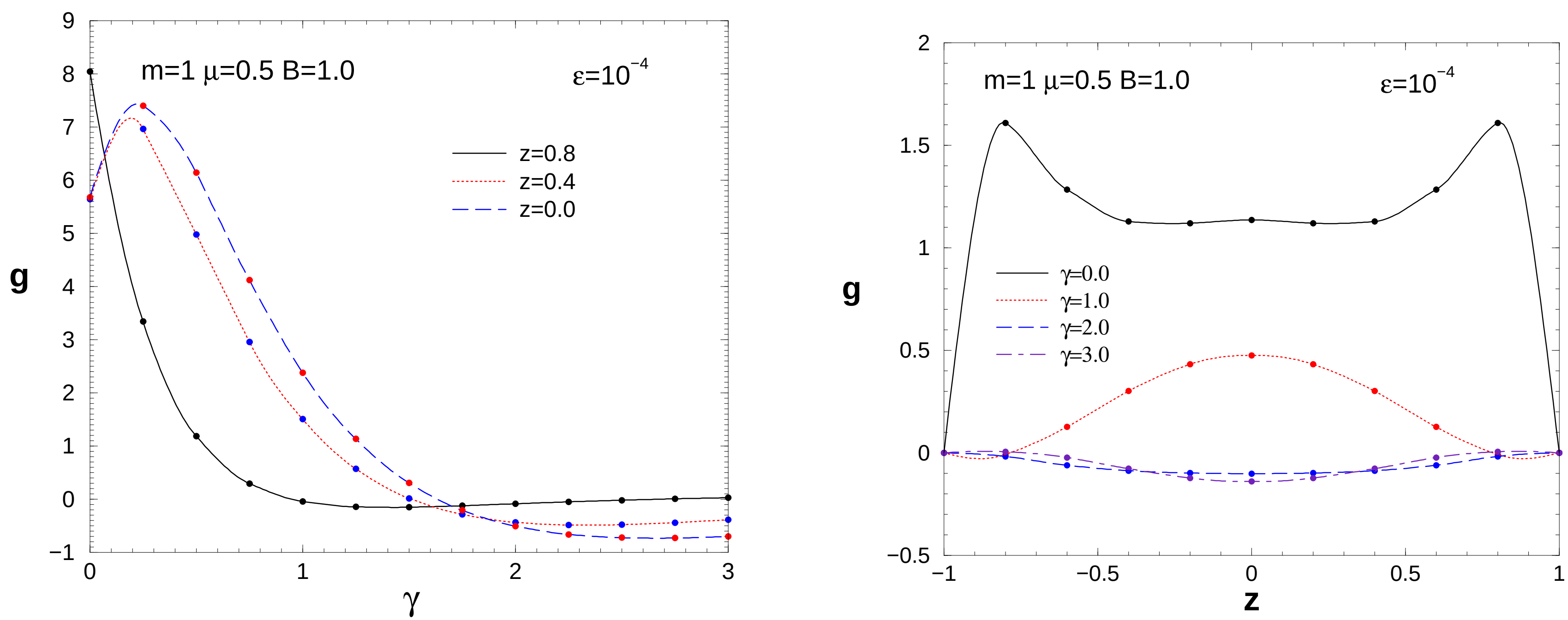
Coupling constant $\alpha = \frac{g^2}{16\pi m^2}$ as a function of the binding energy for $\mu = 0.15$ and $\mu = 0.5$

B	$\alpha(\mu = 0.15)$	$\alpha(\mu = 0.50)$
0.01	0.5716	1.440
0.10	1.437	2.498
0.20	2.100	3.251
0.50	3.611	4.901
1.00	5.315	6.712

These results, **with all shown digits**, coincide with ones obtained in Euclidean space (by Wick rotation).

• This is a test of the method.

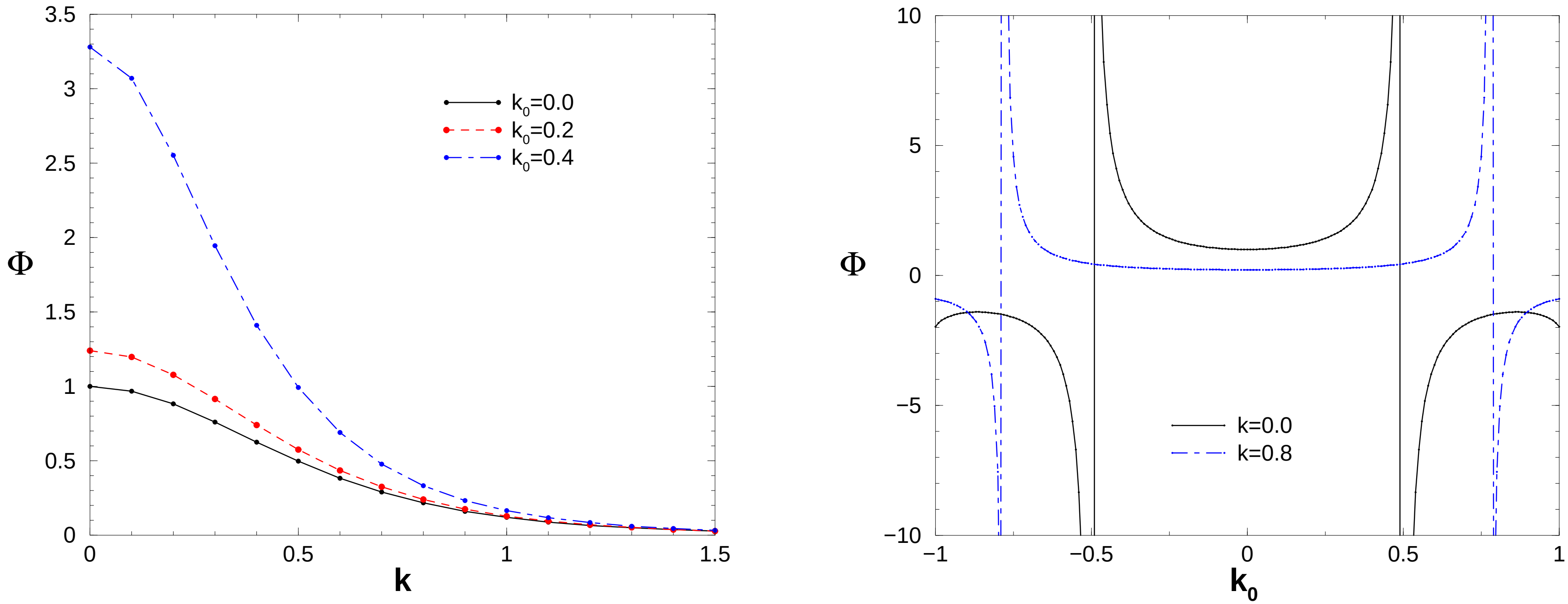
• Function $g(\gamma, z)$



Function $g(\gamma, z)$ for $\mu = 0.5$ and $B = 1.0$. On left – versus γ for fixed values of z and on right – versus z for a fixed values of γ.

• BS amplitude $\Phi(k_0, k)$, $\vec{p} = 0$

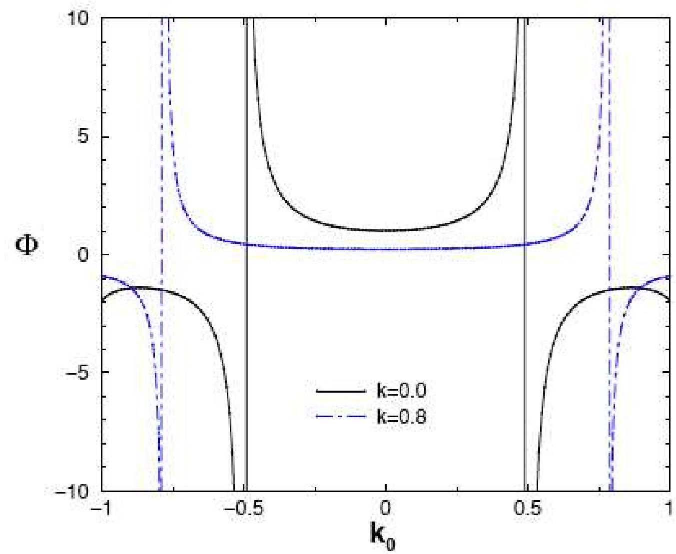
in Minkowski space

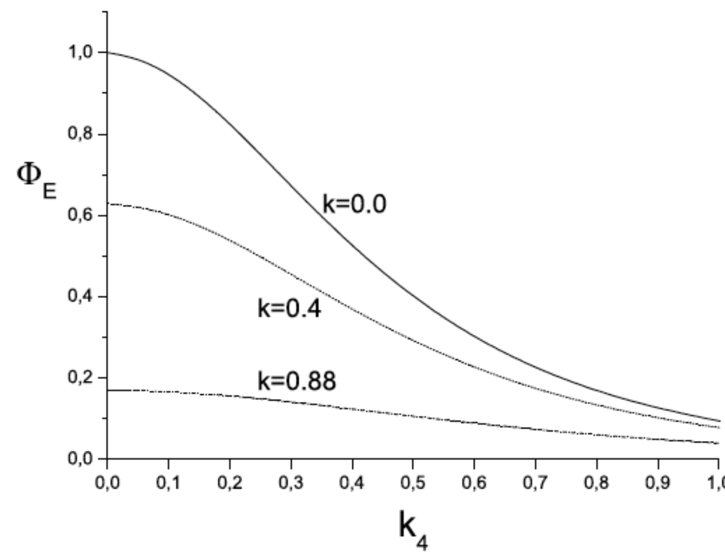


Left: BS amplitude $\Phi(k_0, k)$ vs. k for a fixed values of k_0.
Right: BS amplitude $\Phi(k_0, k)$ vs. k_0 for a fixed values of k.

• BS amplitude

Comparison of Minkowski and Euclidean spaces



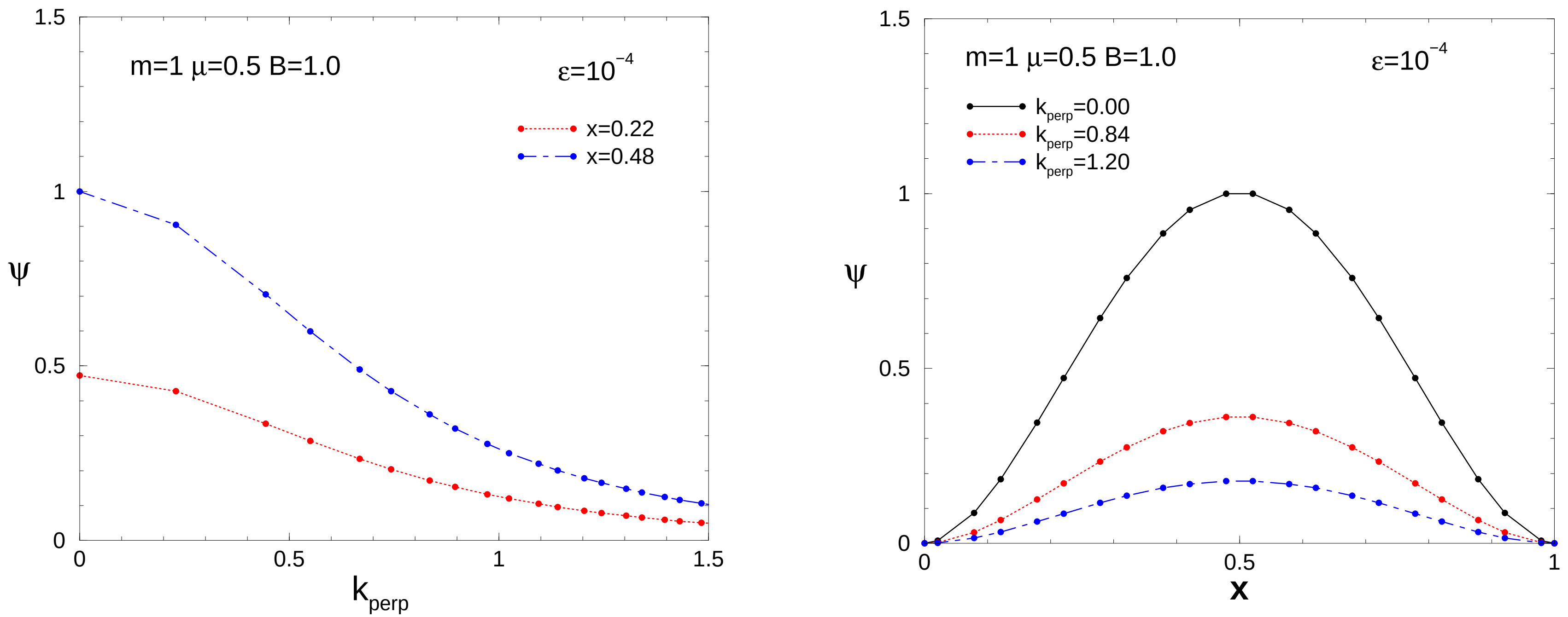


Left: BS amplitude $\Phi(k_0, k)$ in Minkowski space.
Right: BS amplitude $\Phi_E(k_4, k)$ in Euclidean space.

Continuation of Minkowski $\Rightarrow$ Euclidean space exactly coincides with direct solution in Euclidean space.

• LF wave function $\psi(k_\perp, x)$

$$\psi(\vec{k}_\perp, x) = \int_0^\infty \frac{g(\gamma', 1-2x)d\gamma'}{\left[\gamma' + \vec{k}_\perp^2 + m^2 - x(1-x)M^2\right]^2}$$



Left: LFWF $\psi(k_\perp, x)$ versus $k_\perp$ for fixed values of x.

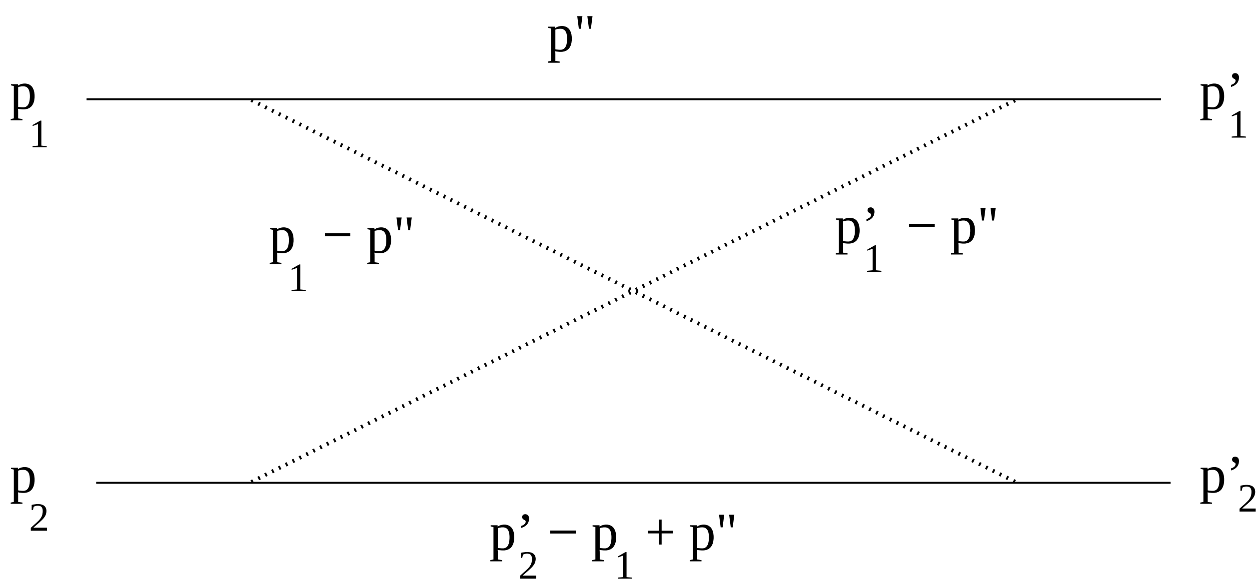
Right: $\psi(k_\perp, x)$ versus x for a few fixed values of $k_\perp$.

• Cross-ladder kernel

Euclidean space: *M.J. Levine and J. Wright, Phys. Rev.* **D2**, *2509 (1970); J.R. Cooke and G.A. Miller, Phys. Rev.* **C62**, *054008 (2000). A. Amghar, B. Desplanques and L. Theusl, Nucl. Phys.* **A 694** *(2001) 439.*

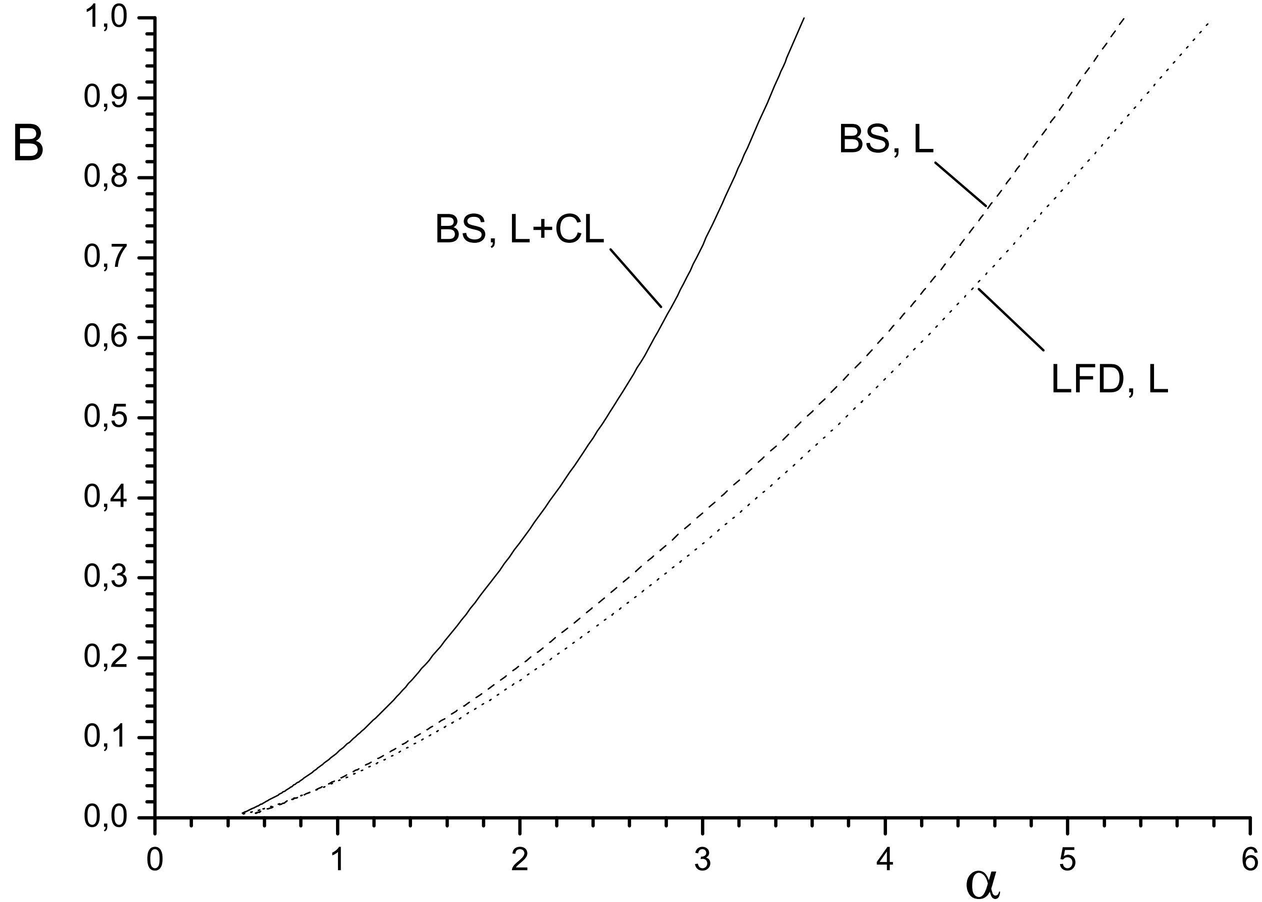
Minkowski space solution:

J. Carbonell and V. A. Karmanov, Eur. Phys. J. **A27** *(2006) 11*



Feynman cross ladder

• Numerical results (L +CL), $\mu = 0.15$



Binding energy B vs. coupling constant α for BS and LFD equations with the ladder (L) kernels only and with the ladder +cross-ladder (L+CL) one for exchange mass $\mu = 0.15$.

• Two fermions

This is much more realistic case.
BS amplitude depends of two spin indices of fermions. It is 2×2 matrix. Decompose it in terms of a basis:

$$\Phi(k,p) = (S_1\phi_1 + S_2\phi_2 + S_3\phi_3 + S_4\phi_4)$$

where

$$S_1 = \gamma_5, \quad S_2 = \frac{1}{M}\hat{p}\,\gamma_5, \quad S_3 = \frac{k{\cdot}p}{M^3}\hat{p}\,\gamma_5 - \frac{1}{M}\hat{k}\,\gamma_5,$$
$$S_4 = \frac{i}{M^2}\sigma_{\mu\nu}p_\mu k_\nu\,\gamma_5$$

with

$$\hat{p} = p_\mu\gamma^\mu, \quad \sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$$

Four scalar functions $\phi_{1-4}(k,p)$.

Nakanishi representation for all components ϕ_i.

$$\begin{aligned}\phi_i(k,p) &= \frac{-i}{\sqrt{4\pi}}\int_{-1}^{1} dz' \\ &\times \int_0^\infty d\gamma' \frac{g_i(\gamma',z')}{\left[\gamma' + m^2 - \frac{1}{4}M^2 - k^2 - p\cdot k\ z' - i\epsilon\right]^3}.\end{aligned}$$

• System of equations

$$\int_0^\infty \frac{g_i(\gamma', z)d\gamma'}{\left[\gamma' + \gamma + z^2m^2 + (1-z^2)\kappa^2\right]^2} =$$

$$\sum_{j=1,2,3,4} \int_0^\infty d\gamma' \int_{-1}^1 dz'\, V_{ij}(\gamma, z; \gamma', z')g_j(\gamma', z')$$

The 4×4 kernel matrix is calculated similarly to the spinless case.

• Meson exchange Lagrangians

Scalar meson exchange Lagrangian:

$$\mathcal{L}^{int} = g_s \; \bar{\psi}\psi\phi^{(s)}$$

Pseudoscalar meson exchange Lagrangian:

$$\mathcal{L}^{int} = i \; g_{ps} \; \bar{\psi}\gamma_5\psi \; \phi^{(ps)}$$

Positronium

$$\mathcal{L}^{int} = g_v\bar{\psi}\gamma^{\mu}\psi A_{\mu}$$

Vertex form factor:

$$F(q) = \frac{\Lambda^2 - \mu^2}{\Lambda^2 - q^2}$$

• Numerical results

Euclidean solution: S.M. Dorkin, M. Beyer, S.S. Semykh and L.P. Kaptari, Few-Body Systems, **42**, 1, (2008).

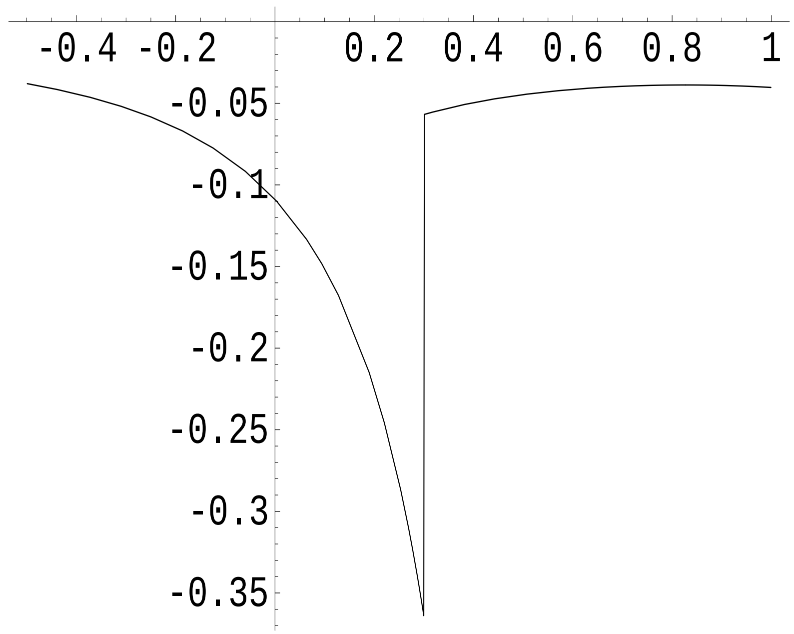
Scalar exchange (Yukawa model)

$\mu = 0.15,\ \Lambda = 2$

B	g^2 (Dorkin et al.)	g^2 (We, Eucl.)	g^2 (We, Mink.)
0.08104	20.23	20.23	20.7
0.14773	30.34	30.34	31.7
0.27765	50.57	50.57	52.15

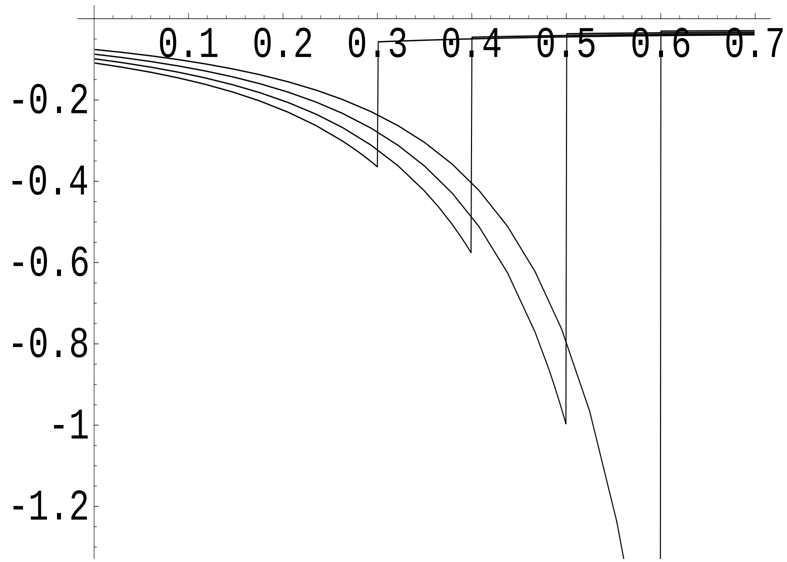
Binding energies, found via Mink. and Euclid, coincide within 2%. Not enough precision, after 4 digits coincidence in the spinless case!

• Discontinuity of $V_{ij}(\gamma, z; \gamma', z')$



One of the matrix elements V_{ij} at $z = 0.3$ v.s. z'. One can see the discontinuity at $z' = z$.

• Discontinuity of $V_{ij}(\gamma, z; \gamma', z')$



Family of matrix elements at $z = 0.3, 0.4, 0.5, 0.6$ v.s. z'.

No catastrophe, but we should take care, choosing a method of the z' integration.

• Improving the method

- Take the BS equation and multiply both sides by $\eta(k,p)$:

$$\eta(k,p)\;\Phi(k,p)=\frac{-i\eta(k,p)}{\left((\frac{p}{2}+k)^2-m^2+i\epsilon\right)\left((\frac{p}{2}-k)^2-m^2+i\epsilon\right)}\int\frac{d^4k'}{(2\pi)^4}K(k,k',p)\Phi(k',p)$$

where

$$\begin{aligned}\eta(k,p) &= \frac{(m^2-L^2)}{\left(k_1^2-L^2+i\epsilon\right)}\frac{(m^2-L^2)}{\left(k_2^2-L^2+i\epsilon\right)}\\ &= \frac{(m^2-L^2)}{\left((\frac{p}{2}+k)^2-L^2+i\epsilon\right)}\frac{(m^2-L^2)}{\left((\frac{p}{2}-k)^2-L^2+i\epsilon\right)}\end{aligned}$$

Equation, before LF projection, remains the same!

- Use Nakanishi representation and apply to both sides the LF projection $\int \ldots dk_-$.
- Obtain new equation for $g(\gamma,z)$. L appears in the equation, but the result does not depend on it!

• New equation for $g(\gamma, z)$

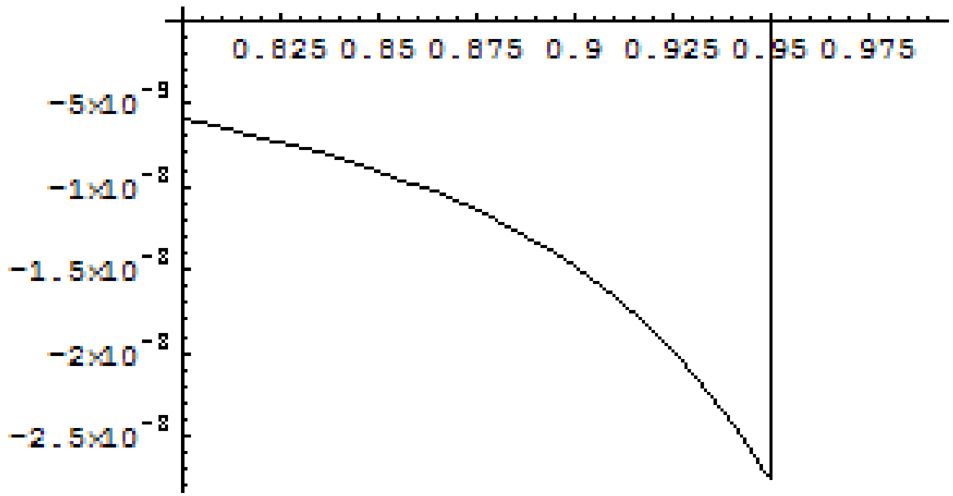
$$\int_0^\infty d\gamma' \int_{-1}^1 dz' F(\gamma, z; \gamma', z')\, g_i(\gamma', z') = \int_0^\infty d\gamma' \int_{-1}^1 dz' \sum_{ij} V_{ij}(\gamma, z; \gamma', z') g_j(\gamma', z')$$

L.-h. side: $F(\gamma, z; \gamma', z')$ – new

R.-h. side: $V_{ij}(\gamma, z; \gamma', z')$ – new

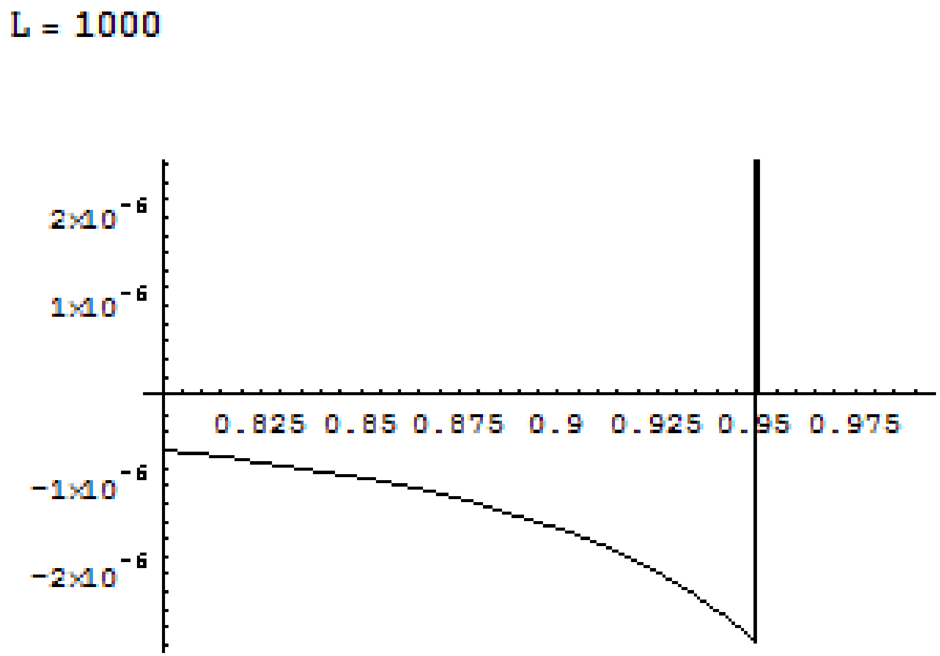
Double integral in l.-h. side.

• **Kernel** $V_{14}(\gamma, z; \gamma', z')$ **v.s.** z'**,** $L = 10000$



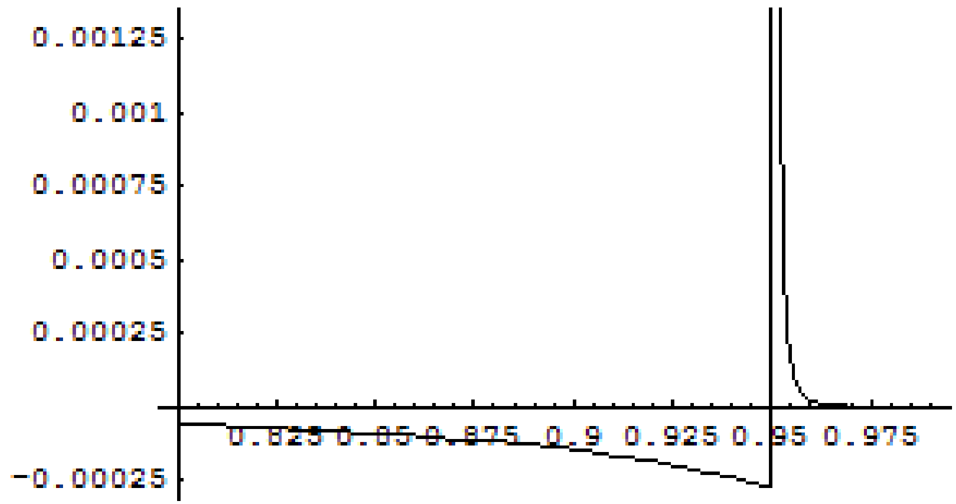
Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 10000$

• Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z', $L = 1000$



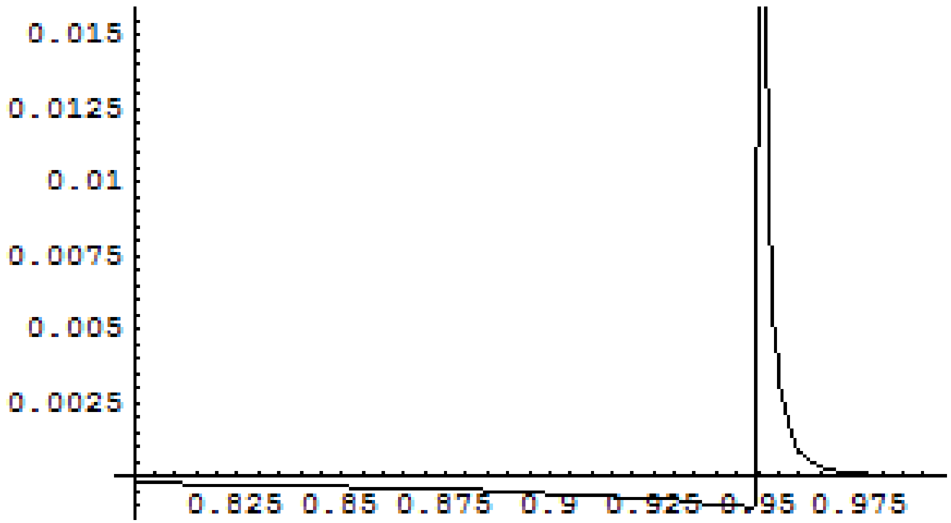
Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 1000$

• **Kernel** $V_{14}(\gamma, z; \gamma', z')$ **v.s.** z'**,** $L = 100$



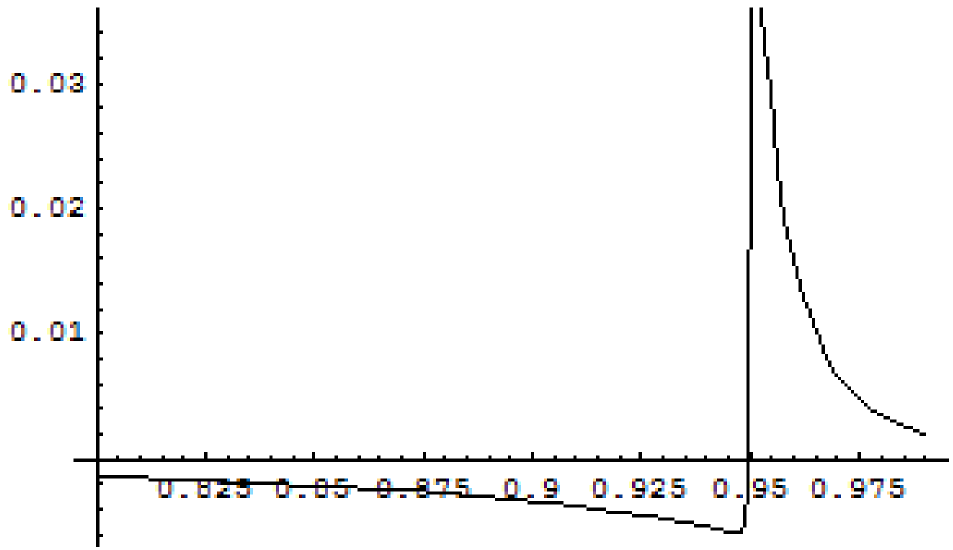
Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 100$

• **Kernel** $V_{14}(\gamma, z; \gamma', z')$ **v.s.** z'**,** $L = 50$



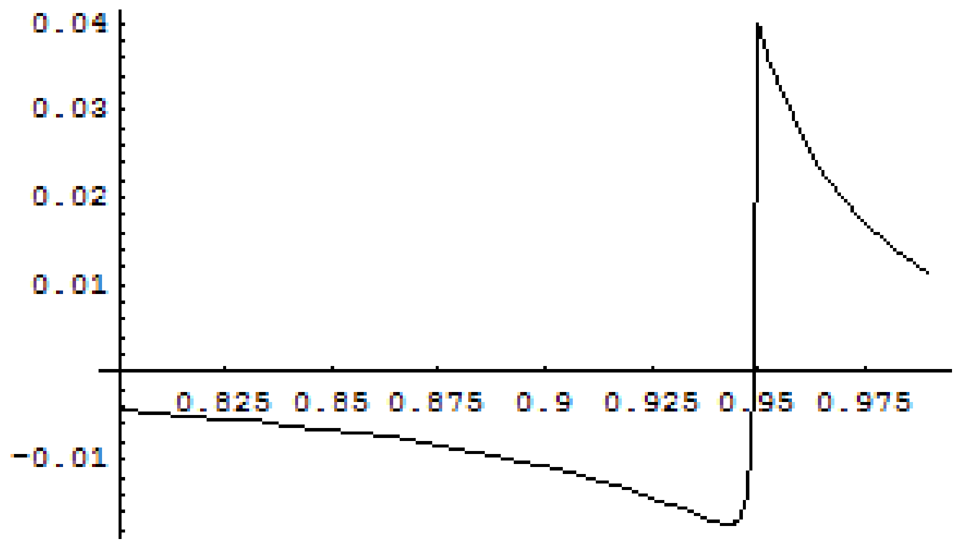
Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 50$

• **Kernel** $V_{14}(\gamma, z; \gamma', z')$ **v.s.** z'**,** $L = 20$



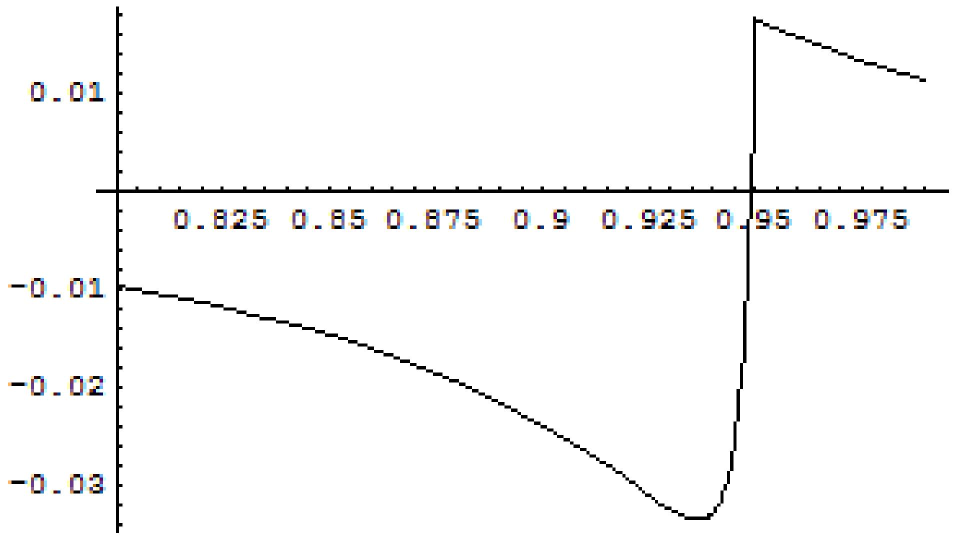
Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 20$

• **Kernel** $V_{14}(\gamma, z; \gamma', z')$ **v.s.** z'**,** $L = 10$



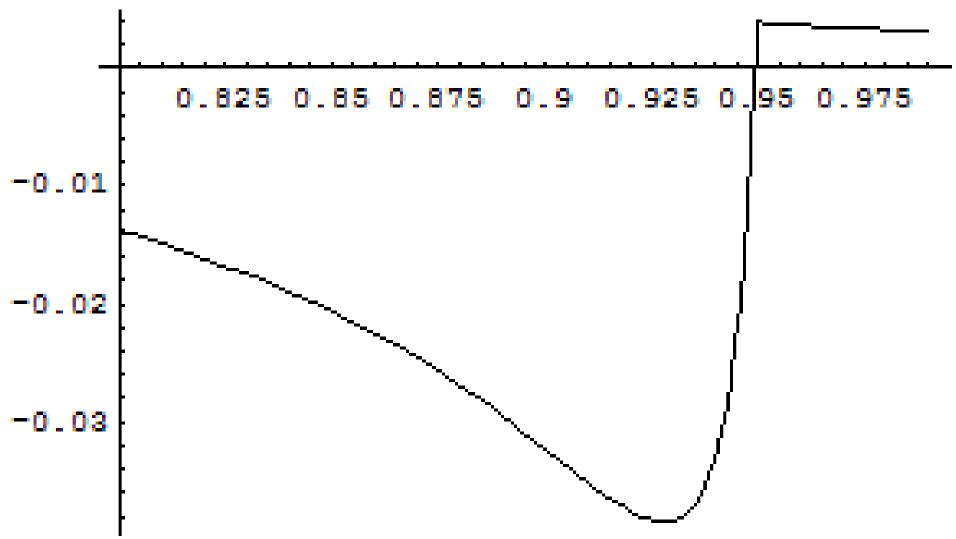
Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 10$

• **Kernel** $V_{14}(\gamma, z; \gamma', z')$ **v.s.** z'**,** $L = 5$



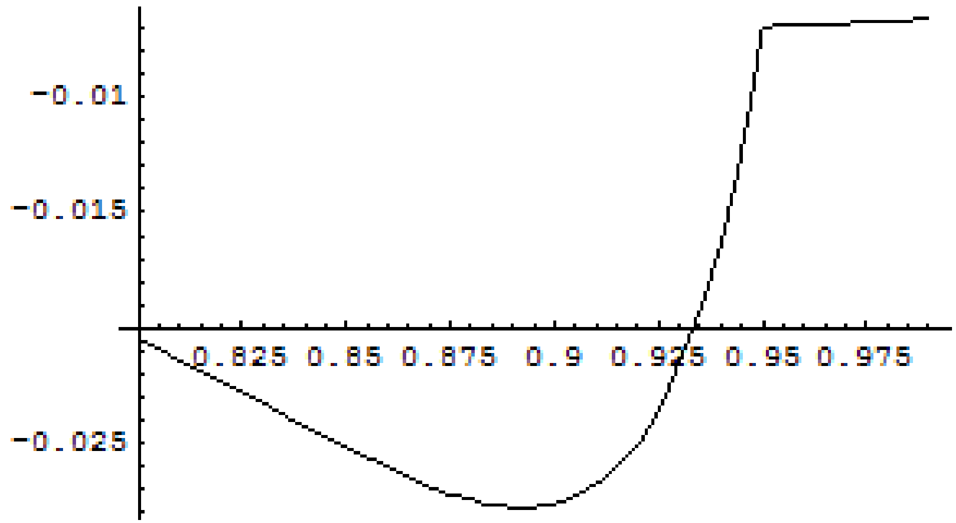
Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 5$

• Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z', $L = 3$



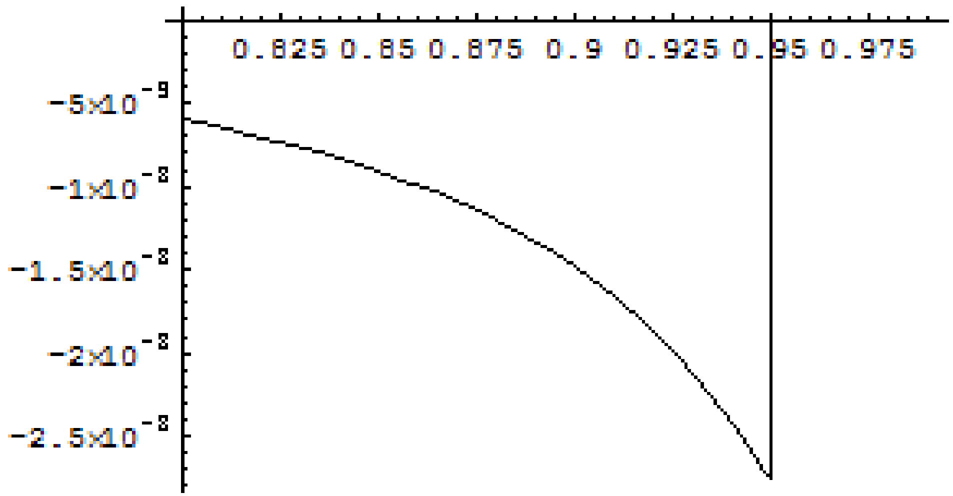
Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 3$

• **Kernel** $V_{14}(\gamma, z; \gamma', z')$ **v.s.** z'**,** $L = 1.1$



Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 1.1$

- **Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z', $L = 10000$**



Kernel $V_{14}(\gamma, z; \gamma', z')$ v.s. z' for fixed $z = 0.95$, $L = 10000$

• Numerical results

Scalar exchange (Yukawa model)

$\mu = 0.15$, $\Lambda = 2$, $L = 1.1$

B	g^2 (Dorkin et al.)	g^2 (We, Eucl.)	g^2 (We, Mink.)
0.08104	20.23	20.23	20.23
0.14773	30.34	30.34	30.34
0.27765	50.57	50.57	50.57

Binding energies, found via Mink. and Euclid, coincide now within 4 digits. Good precision!

• Numerical results

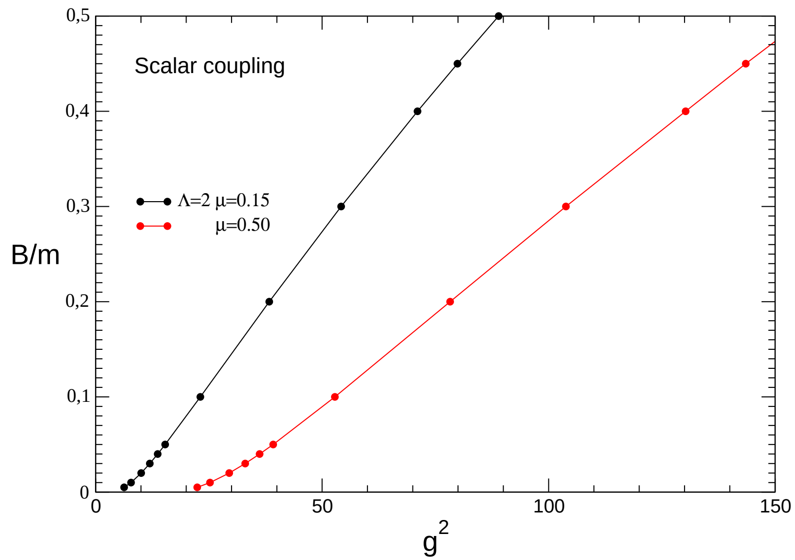
Pseudo scalar exchange

$\mu = 0.15,$, $\Lambda = 2$, $L = 1.1$

B	g^2 (Dorkin et al.)	g^2 (We, Eucl.)	g^2 (We, Mink.)
0.1	260.8	262.1	262.1

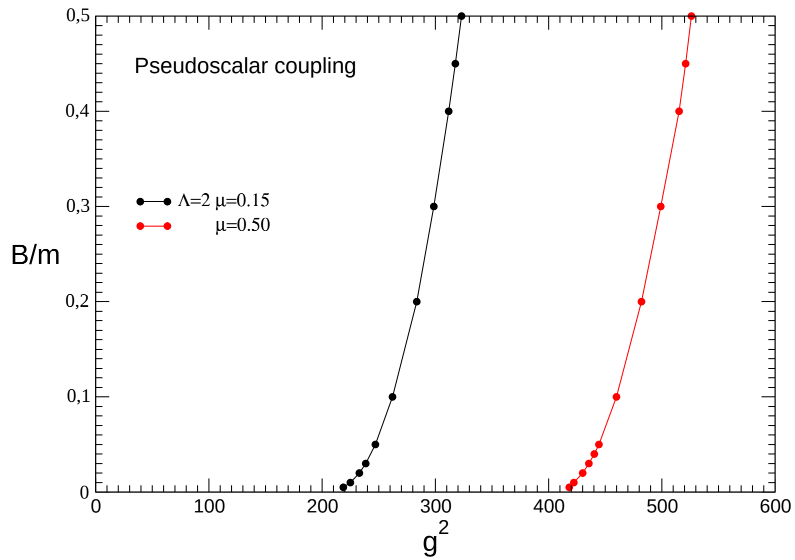
Our binding energies, found via Mink. and Euclid, coincide within 4 digits. Difference with Dorkin et al. is 0.5%. Good precision!.

Binding energy for scalar exchange v.s. g



Binding energy for scalar exchange v.s. g^2 for $\Lambda = 2$, $L = 1.1$, $\mu = 0.15$ and $\mu = 0.5$

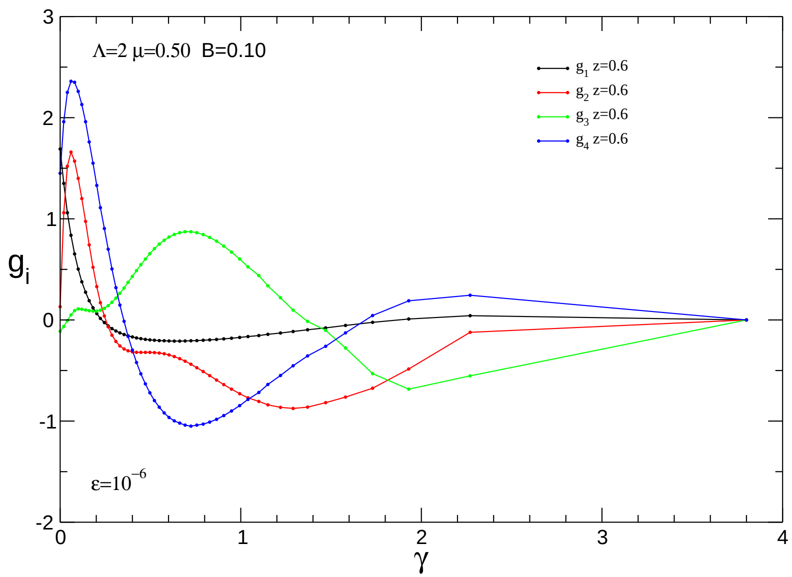
• Binding energy for PS exchange v.s. g^2

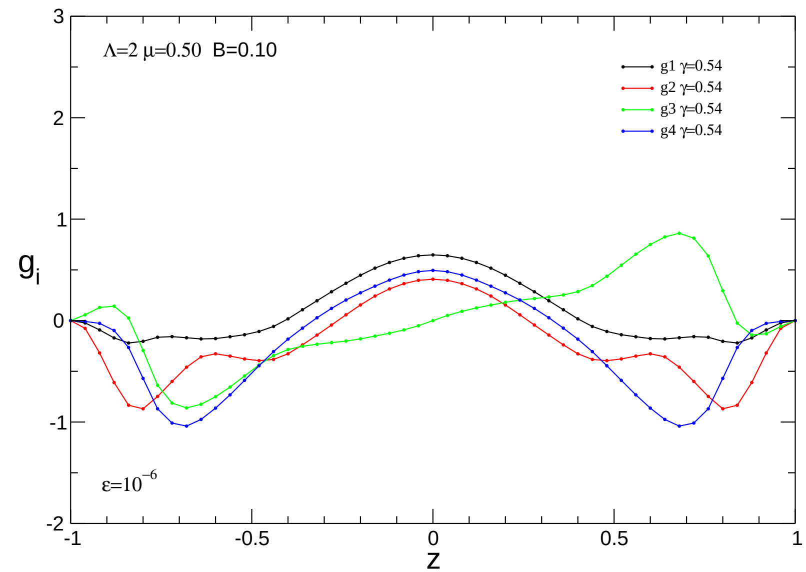


Binding energy for pseudo scalar exchange v.s. g^2 for $\Lambda = 2$, $L = 1.1$, $\mu = 0.15$ and $\mu = 0.5$

• Weight functions $g_i(\gamma, z)$

Scalar exchange



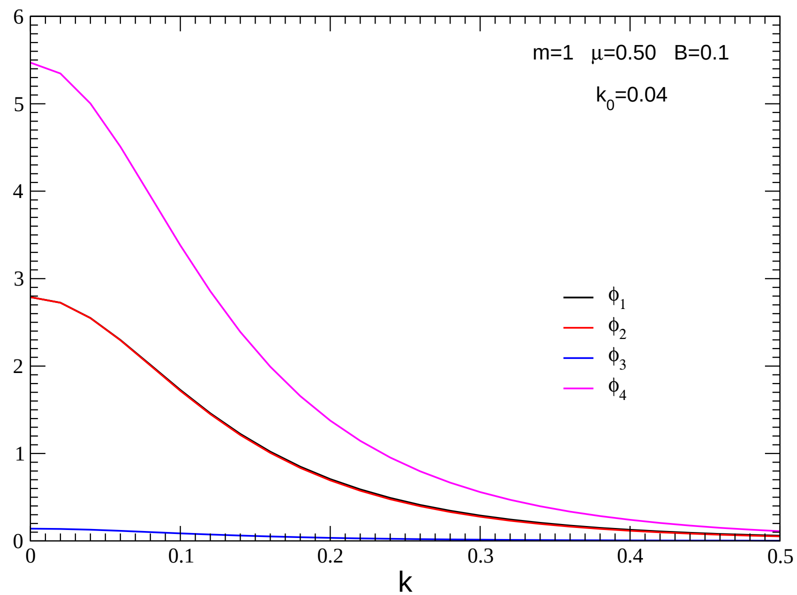


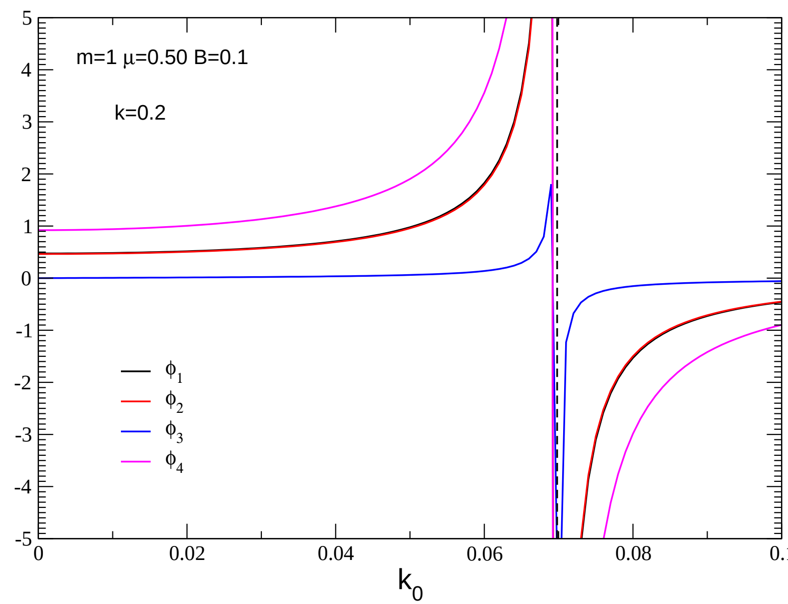
Left: Nakanishi weight functions v.s. γ for $z = 0.6$, for scalar exchange for $\Lambda = 2$, $L = 1.1$, $\mu = 0.15$ and $\mu = 0.5$.

Right: Nakanishi weight functions v.s. z for $\gamma = 0.54$.

• BS amplitudes $\Phi_i(k_0, k)$, $\vec{p} = 0$

In Minkowski space

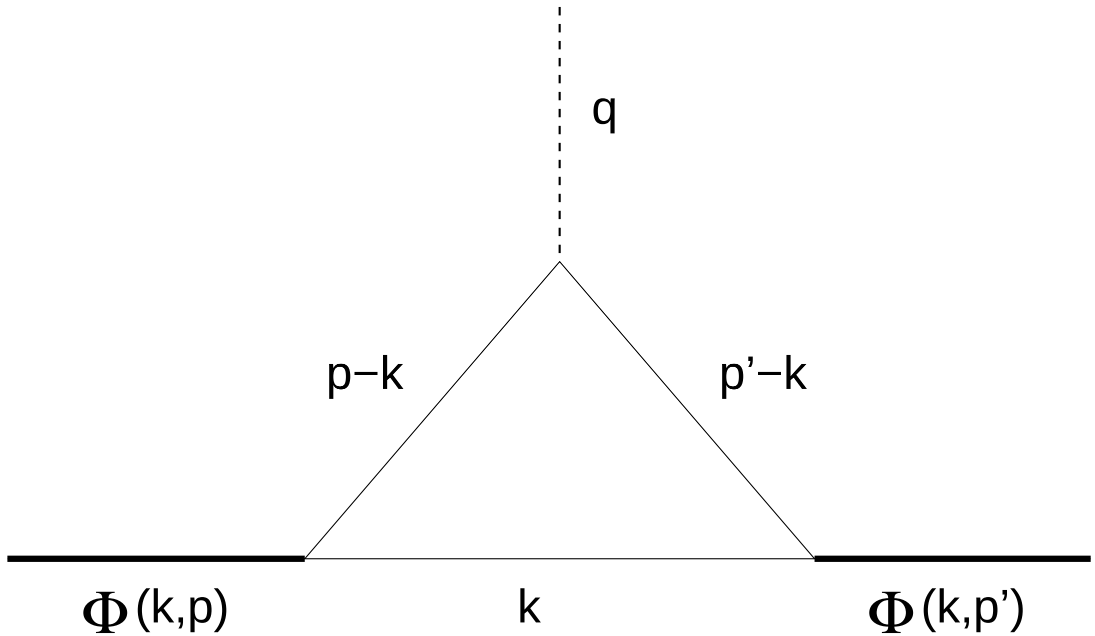




Left: Minkowski BS amplitudes vs. $k = |\vec{k}|$ for $k_0 = 0.04$. The amplitudes ϕ_1 and ϕ_2 are indistinguishable.

Right: The same vs. k_0 for $k = |\vec{k}| = 0.2$

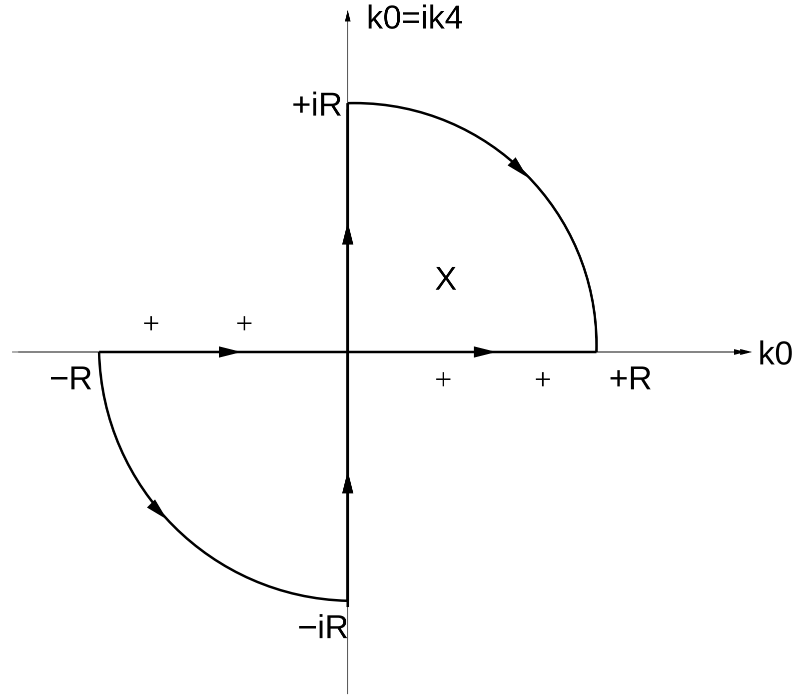
• Profit: EM form factor



E.m. vertex in terms of the BS amplitude.

• FF via Euclidean BS solution

Impossibility of Wick rotation.



Wick rotation in the form factor integral.

The singularities X in the first quadrant prevent from the Wick rotation!

Complex boost

$$\Phi(k,p) = \Phi(k^2, k\cdot p)$$

$$k\cdot p = k_0 p_0 - \vec{k}\vec{p} \quad \Rightarrow \quad ik_4 p_0 - \vec{k}\vec{p}$$

However, solvable numerically (P. Maris et al.)

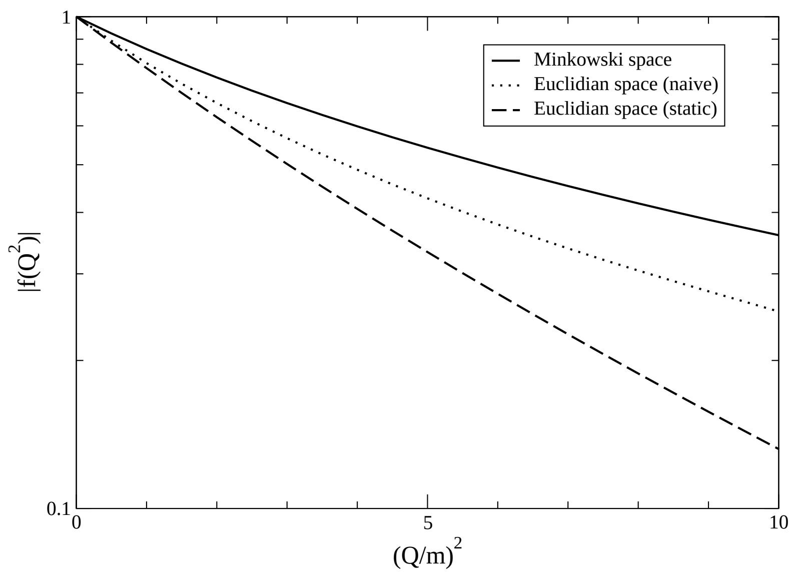
Static approximation, to avoid complex boost (M.J. Zuilhof and J.A. Tjon):

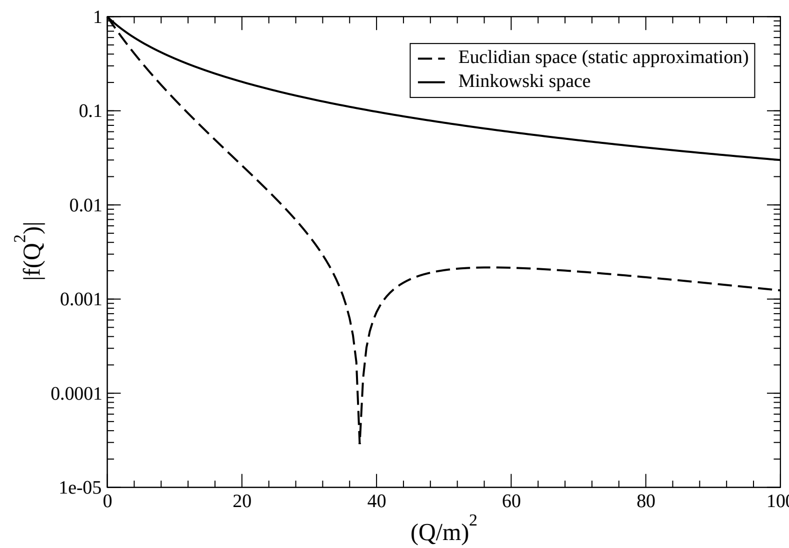
Take Euclidean BS amplitude at rest: $\Phi_E(k_4, \vec{k})$

Boost only the spatial component:

$$\Phi_E(k_4, \vec{k}) \quad \Rightarrow \quad \Phi_E(k_4, \vec{k} + \vec{p}')$$

• Calculating EM form factors



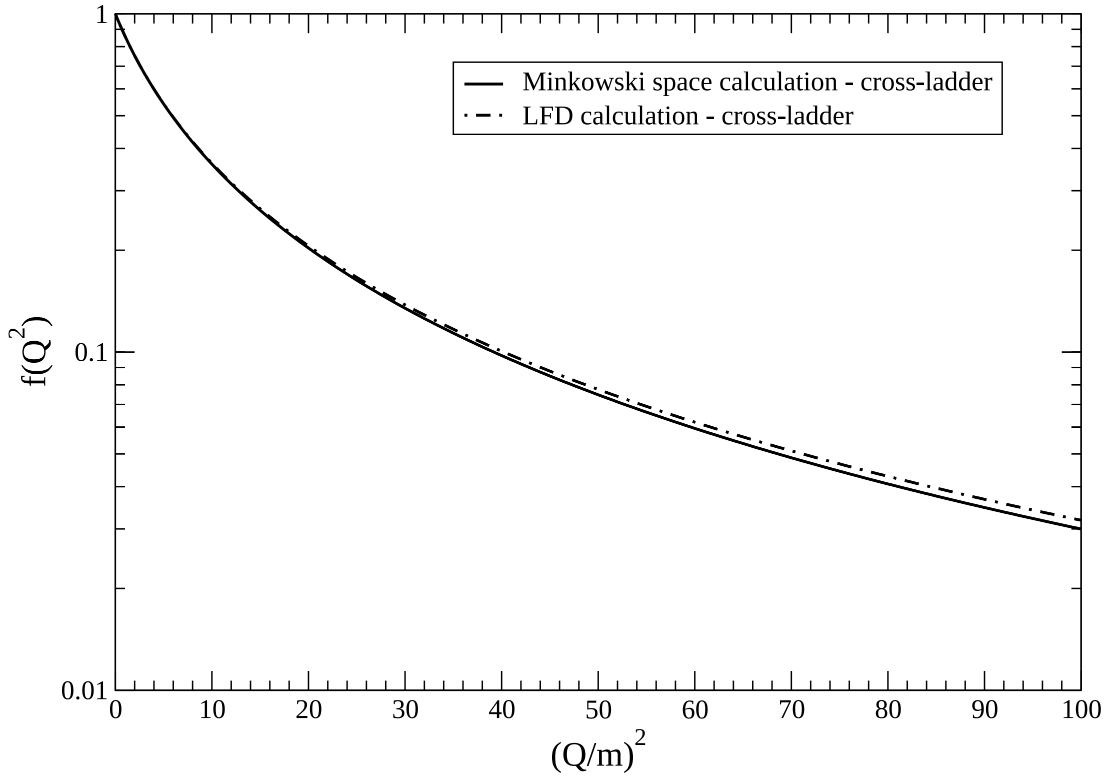


Left: Form factor via Minkowski BS amplitude (solid curve), Euclidean one and and Euclidean one in static approximation (dashed).

Right: Form factor via Minkowski BS amplitude (solid curve, the same as at left panel) and in LFD (dot-dashed).

With BS solution in Minkowski space we can avoid all these problems and approximations!

• Minkowski vs. light-front solutions



Form factor via Minkowski BS amplitude (solid curve, the same as at left panel) and in LFD (dot-dashed).

• Conclusions

- A method to find Bethe-Salpeter amplitude in Minkowski space, both for spinless particles and fermions, is developed.
- The method is applied to the ladder (OBE) kernel and to the ladder +cross-ladder one.
- It can be applied to any kernel given by a Feynman graphs.
- It gives, as a by-product, LF wave function.
- Minkowski BS amplitude allows to calculate observables without uncontrolled approximations.