Bound states in QED

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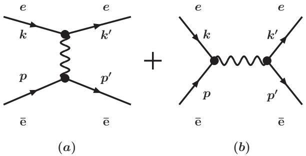
- Bound states is not correctly formulated problem in QFT
- Standard approaches $\Rightarrow$ reduce to non-relativistic Schr.-eq.
  1. Bethe-Salpeter eq., Quasipotential eq.;
  2. Effective non-relativistic Lagrangians
- TIME should be excluded to get potential picture.
- Relativistic QFT $\Rightarrow$ Functional integral approach
  1. Formally exact representation for any Green functions;;
  2. Standard using $\Rightarrow$ representation of perturbation series.
- Our results
  1. Definition and calculation of bound states.
  2. Role of TIME in bound states formation
- Relativistic QED
  1. Bound states does exist
  2. Role of TIME is important $\Rightarrow$ corrections of $\alpha^{\frac{2}{3}}$;
  3. orth0- para- positronium mass difference is not described correctly.
- QED is not suited to describe bound state problem correctly

Breit potential - electron+positron



♣*Born approximation* $\mathbf{k} \to 0 \quad A(\mathbf{k}) \sim \tilde{U}(\mathbf{k}) \quad \Rightarrow \quad U(r)$

♣ *e and ē* $\to$ *Solutions of the Dirac equation.*

$$U_a(r) = -\frac{\alpha}{r} + \frac{2\pi}{3}\frac{\alpha}{m^2}(\boldsymbol{\sigma}_-\boldsymbol{\sigma}_+)\delta(\mathbf{r}), \qquad U_b(r) = \frac{\pi}{2}\frac{\alpha}{m^2}(\boldsymbol{\sigma}_-\boldsymbol{\sigma}_+)\delta(\mathbf{r}),$$

$$\Delta = \epsilon_{ortho} - \epsilon_{para} = \frac{7}{12}\alpha^4 m = \left[\left(\frac{1}{3}\right)_{(a)} + \left(\frac{1}{4}\right)_{(b)}\right]\alpha^4 m$$

$$\Delta_{exp} = 203.38910\ GHz = 8.4115 \cdot 10^{-4}\ eV = \frac{7}{12}\alpha^4 m_e \cdot 0.99512...$$

QFT

$$\textit{Current} \qquad \mathbf{J}_\Gamma(x) = (\overline{\psi}(x)\Gamma\psi(x))$$

$$G_\Gamma(x) = \langle 0|\mathbf{J}_\Gamma(x)\mathbf{J}_\Gamma(0)|0\rangle$$

$$= \sum_{particles} \langle 0|\mathbf{J}_\Gamma(x)|n\rangle\langle n|\mathbf{J}_\Gamma(0)|0\rangle + \sum_{photons} \langle 0|\mathbf{J}_\Gamma(x)|n\rangle\langle n|\mathbf{J}_\Gamma(0)|0\rangle$$

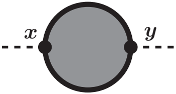
$$= \sum_{particles} e^{-E_n|x|}|\langle 0|\mathbf{J}_\Gamma(0)|n\rangle|^2 + \sum_{photons} e^{-E_n|x|}|\langle 0|\mathbf{J}_\Gamma(0)|n\rangle|^2$$

$$\Longrightarrow e^{-E_{min}|x|} + \frac{1}{|x|^2}, \qquad |x| \to \infty$$

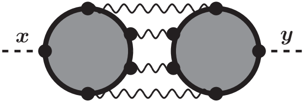
QED: Lagrangian and bound states

$$\mathcal{L}(x) = -\frac{1}{4}\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right)^2 + \left(\overline{\psi}(x)[i(\hat{p} + e\hat{A}(x)) - m]\psi(x)\right)$$

$$G_\Gamma(x-y) = \langle 0|\mathbf{J}_\Gamma(x)\mathbf{J}_\Gamma(y)|0\rangle = \int DAD\psi D\overline{\psi}\; e^{\int dx \mathcal{L}(x)}$$



$\mathrm{B}(x-y) \sim e^{-M|x-y|}$



$\mathrm{H}(x-y) \sim \frac{1}{|x-y|^2}$

Mass of the bound state $M = -\lim_{|x|\to\infty} \frac{1}{|x|} \ln \mathbf{B}(x)$

$$\mathbf{B}_\Gamma(x) = \int \frac{DA}{C} e^{-\frac{1}{2}(A_\mu D^{-1}_{\mu\nu} A_\nu)} \cdot \mathrm{Tr}[\Gamma S(x,0|A)\Gamma S(0,x|A)]$$

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$$D_{\mu\nu}(k) = \frac{1}{k^2}\left[\delta_{\mu\nu} + d(k^2)\frac{k_\mu k_\nu}{k^2}\right]$$

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$$S(x,0|A) = \frac{1}{i(\hat{p} + e\hat{A}(x)) - m}\delta(x) \qquad (X = |x| \to \infty)$$

$$\Rightarrow \frac{\mathrm{const}}{X^{\frac{1}{2}}}(1+\gamma_0)e^{-mX} \cdot \mathcal{S}(X|A); \qquad (z_\mu(\tau) = n_\mu \tau + \eta_\mu(\tau))$$

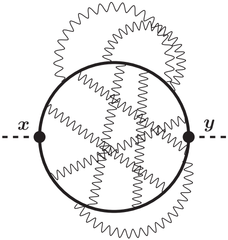
$$\mathcal{S}(X|A) = \int \frac{D\eta}{C} e^{-\int_0^X d\tau \left[\frac{m\dot{\eta}^2(\tau)}{2} + ie\dot{z}_\mu(\tau)A_\mu(z(\tau))\right]} R[z],$$

$$R[z] = \mathrm{T}_\tau\left\{e^{\frac{e}{4m}\int_0^X d\tau\ \sigma_{\mu\nu}(\tau)F_{\mu\nu}(z(\tau))}\right\}$$

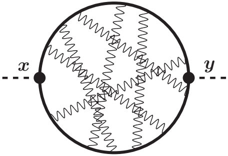
$$\mathbf{B}_\Gamma(X) \sim e^{-2mX} \iint \frac{D\eta_1 D\eta_2}{C} e^{-\int\limits_0^X d\tau \frac{m}{2}\left[\dot{\eta_1}^2(\tau)+\dot{\eta}_2^2(\tau)\right]}$$

$$\cdot \int \frac{DA}{C} e^{-\frac{1}{2}(A_\mu D^{-1}_{\mu\nu} A_\nu) + ie\int\limits_0^x d\tau \dot{z}^{(1)}_\mu(\tau) A_\mu(z^{(1)}(\tau)) + ie\int\limits_0^X d\tau \dot{z}^{(2)}_\mu(\tau) A_\mu(z^{(2)}(\tau))}$$

$$\cdot \frac{1}{4} \mathrm{Tr}\left[\Gamma \ \ (1+\gamma_0)\, e^{\frac{e}{4m}(\sigma_{\mu\nu} F_{\mu\nu}[z^{(1)}])} \ \Gamma \ \ (1-\gamma_0)\, e^{\frac{e}{4m}(\sigma_{\mu\nu} F_{\mu\nu}[z^{(2)}])}\right]$$



$\Longrightarrow$



$$\frac{1}{4}\mathrm{Tr}\left[\Gamma\ \ (1+\gamma_0)\, e^{\frac{e}{4m}(\sigma_{\mu\nu}F_{\mu\nu}[z^{(1)}])}\ \Gamma\ \ (1-\gamma_0)\, e^{\frac{e}{4m}(\sigma_{\mu\nu}F_{\mu\nu}[z^{(2)}])}\right]$$

$$\Longrightarrow \Sigma_\Gamma^{(0)} \cdot e^{\frac{e^2}{(4m)^2}\int dk\mathcal{K}(k)\Delta_\Gamma(k) + O(e^4)}$$

J^P	Γ	$\Sigma_\Gamma^{(0)}$	$\Delta_\Gamma(k)$
$P(0^-)$	$i\gamma_5$	-2	$-2 \cdot \frac{\mathbf{k}^2}{\mathbf{k}^2+k_4^2}$
$V(1^-)$	γ_j	$-2\delta_{ij}$	$\frac{2}{3} \cdot \frac{\mathbf{k}^2}{\mathbf{k}^2+k_4^2}$
$S(0^+)$	I	0	0
$A((1^+)$	$\gamma_5\boldsymbol{\gamma}$	0	0

Functional integral problem

$$d\sigma_{\eta_1\eta_2} = \frac{D\eta_1 D\eta_2}{C} e^{-\int\limits_0^X dt\, \frac{m}{2}\left[\dot{\eta_1}^2(t)+\dot{\eta}_2^2(t)\right]+W}, \qquad J_0(X) = \int d\sigma_{\eta_1\eta_2} = J_0 e^{\epsilon_0 X}$$

$$W[X;\eta_1,\eta_2] = e^2 \iint\limits_0^X d\tau_1 d\tau_2 \int \frac{dk}{(2\pi)^4} \cdot \frac{e^{ik(\tau_1-\tau_2)}}{k^2} e^{ik(\eta_1(\tau_1)-\eta_2(\tau_2))}$$

$$\epsilon_\Gamma = \lim_{X\to\infty} \frac{e^2}{4m^2} \frac{1}{X} \iint\limits_0^X d\tau_1 d\tau_2 \int \frac{dk\Delta_\Gamma(k) e^{ik_4(\tau_1-\tau_2)}}{(2\pi)^4} \cdot \left\langle e^{ik(\eta_1(\tau_1)-\eta_2(\tau_2))} \right\rangle_{\eta_1\eta_2}$$

$$\left\langle e^{ik(\eta_1(\tau_1)-\eta_2(\tau_2))} \right\rangle_{\eta_1\eta_2} = \frac{1}{J_0(X)} \iint d\sigma_{\eta_1\eta_2} e^{ik(\eta_1(\tau_1)-\eta_2(\tau_2))}$$

Functional integral and α

$$Y = \alpha^2 mX, \quad \tau = \frac{v}{\alpha^2 m}, \quad k_4 = \alpha^2 mq, \quad \mathbf{k} = \alpha m\mathbf{q}.$$

$$\boldsymbol{\eta}(t) = \frac{1}{\alpha m}\boldsymbol{\xi}(v), \quad \eta(t) = \frac{1}{\alpha m}\xi(v).$$

$$\boldsymbol{\xi}_{1,2}(\tau) = \mathbf{R}(\tau) \pm \frac{1}{2}\boldsymbol{\rho}(\tau), \qquad \xi_{1,2}(\tau) = R(\tau) \pm \frac{1}{2}\rho(\tau),$$

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$$W[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \xi_1, \xi_2; \alpha]$$

$$= \iint_0^Y dv_1 dv_2 \iint \frac{d\mathbf{q}dq}{4\pi^3} \frac{e^{iq(v_1 - v_2)}}{\mathbf{q}^2 + \alpha^2 q^2} \cdot e^{i\alpha q(\xi_1(v_1) - \xi_2(v_2)) + i\mathbf{q}(\boldsymbol{\xi}_1(v_1) - \boldsymbol{\xi}_2(v_2))}$$

$$\Rightarrow \int_0^Y \frac{dv}{|\boldsymbol{\rho}(v)|} - \frac{\alpha}{8} \int_0^Y dv|\rho(v)| + O(\alpha^2)$$

$$d\sigma = \frac{D\boldsymbol{\xi}_1 D\boldsymbol{\xi}_2 D\xi_1 D\xi_2}{C} e^{-\frac{1}{2}\int_0^Y dv\left[\dot{\boldsymbol{\xi}}_1^2(v)+\dot{\boldsymbol{\xi}}_2^2(v)+\dot{\xi}_1^2(\tau)+\dot{\xi}_2^2(v)\right]+W[\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\xi_1,\xi_2;\alpha]}$$

$$\Longrightarrow \frac{D\boldsymbol{\rho}}{C_{\boldsymbol{\rho}}} e^{-\int_0^{\alpha^2 mX} dv\left[\frac{1}{4}\dot{\boldsymbol{\rho}}^2(v)-\frac{1}{|\boldsymbol{\rho}(v)|}\right]} \cdot \frac{D\rho}{C_\rho} e^{-\int_0^{\alpha^2 mX} dv\left[\frac{1}{4}\dot{\rho}^2(v)+\frac{\alpha}{8}|\rho(v)|\right]}.$$

Coulomb and linear potentials

$$\begin{cases} \left[-\frac{1}{2m}\frac{d^2}{d\mathbf{x}^2}-\frac{\alpha}{|\mathbf{x}|}\right]\Psi = -\epsilon\Psi \\ \mathbf{x} = \frac{\mathbf{y}}{\alpha}, \qquad \epsilon = \varepsilon\alpha^2 \\ \left[-\frac{1}{2m}\frac{d^2}{d\mathbf{y}^2}-\frac{1}{|\mathbf{x}|}\right]\Psi = -\varepsilon\Psi \end{cases}, \qquad \begin{cases} \left[-\frac{d^2}{ds^2}-\alpha|s|\right]\Phi = E\Phi \\ s = \frac{v}{\alpha^{\frac{1}{3}}}, \qquad E = \mathcal{E}\alpha^{\frac{2}{3}} \\ \left[-\frac{d^2}{dv^2}-|v|\right]\Phi = \mathcal{E}\Phi \end{cases}$$

Orto-para- mass difference

$$\delta M = \frac{1}{3} \cdot \alpha^4 m \cdot \Delta(\alpha),$$

$$\Delta(\alpha) = \lim_{Y\to\infty} \frac{1}{Y} \iint\limits_0^Y dv_1 dv_2 \iint \frac{d\mathbf{q}dq}{2\pi^3} \cdot \frac{\mathbf{q}^2 \; e^{iq(v_1-v_2)-\frac{1}{4}[\alpha^2 q^2+\mathbf{q}^2]|v_1-v_2|}}{\mathbf{q}^2+\alpha^2 q^2}$$

$$\cdot \left\langle e^{-\frac{i\mathbf{q}}{2}(\boldsymbol{\rho}(v_1)+\boldsymbol{\rho}(v_2))} \right\rangle_{\boldsymbol{\rho}} \left\langle e^{-\frac{i\alpha q}{2}(\rho(v_1)+\rho(v_2))} \right\rangle_{\rho}$$

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$$\left\langle e^{i\frac{\mathbf{k}}{2}[\boldsymbol{\rho}(\tau_1)+\boldsymbol{\rho}(\tau_2)]} \right\rangle_{\boldsymbol{\rho}} \Rightarrow \sum_{n\ell} e^{-|\tau_1-\tau_2|(E_n-E_{00})}(-1)^{\ell}(2\ell+1)\mathbf{C}^2_{n\ell}\left(\frac{k}{2}\right).$$

$$\left\langle e^{-i\frac{q}{2}(\rho(\tau_1)+\rho(\tau_2))} \right\rangle_{\rho} \Rightarrow \sum_{\kappa} e^{-|\tau_1-\tau_2|(E_\kappa-E_0)}(-1)^{\kappa} \left|\mathcal{A}_\kappa\left(\alpha^{\frac{2}{3}}q\right)\right|^2$$

$$\Delta(\alpha) = \sum_{n=1}^{\infty} \sum_{\ell=0}^{n-1} \sum_{\kappa=0}^{2} (-1)^{\ell+\kappa} (2\ell+1) \frac{32}{\pi^2} \iint_0^{\infty} \frac{dkdq\ k^4}{k^2+\alpha^2 q^2}$$

$$\cdot \frac{k^2+\alpha^2 q^2 + 1 - \frac{1}{n^2} + \alpha^{\frac{2}{3}}(\epsilon_\kappa - \epsilon_0)}{\left(k^2+\alpha^2 q^2 + 1 - \frac{1}{n^2} + \alpha^{\frac{2}{3}}(\epsilon_\kappa - \epsilon_0)\right)^2 + 16q^2} \mathbf{C}_{nl}^2 \left(\frac{k}{2}\right) \left|\mathcal{A}_\kappa \left(\alpha^{\frac{2}{3}} q\right)\right|^2$$

$\diamondsuit$

$$\delta M = \left(\frac{7}{12}\right) \frac{1}{3} \cdot \alpha^4 m \cdot \Delta(\alpha) = \begin{cases} \Delta(0) = 1.0 \\ \Delta(\alpha) = 0.9528 \approx 1 - \frac{5}{4}\alpha^{\frac{2}{3}} \end{cases}$$

$$\Delta_{exp} = 1 - \frac{2}{3}\alpha = 0.9951$$

Conclusion

- The functional approach is the best mathematical representation to preserve the gauge invariance.
- The lowest approximation of this functional representation is pure non-relativistic Feynman representation of the non-relativistic Schrödinger equation.
- Any regular series for next corrections over α do not exist and these corrections can not be reduced to some terms into non-relativistic potential in the Schrödinger picture.
- The "non-physical" time coordinate plays important role in the relativistic QED and leads to nonanalytic corrections of the order $\alpha^{\frac{2}{3}}$.
- Non-relativistic QED and Relativistic QED give the different numbers for ortho- para- mass difference.
- Relativistic QED is not suited to describe real bound states.
- Non-relativistic QED and Relativistic QED are different theories.
- Dirac (1981): Today QFT is not THEORY, but set of rules to calculate amplitudes.

Appendix

$$H = -\left(\frac{\partial}{\partial \mathbf{x}}\right)^2 + U(|\mathbf{x}|), \qquad H\Psi_n(\mathbf{x}) = E_n\Psi_n(\mathbf{x}).$$

$$G_X(\mathbf{x},\mathbf{x}') = e^{-HX}\delta(\mathbf{x}-\mathbf{x}') = \int\limits_{\boldsymbol{\rho}(0)=\mathbf{x}',\boldsymbol{\rho}(X)=\mathbf{x}} \frac{D\boldsymbol{\rho}}{C} e^{-\int_0^X d\tau\left[\frac{1}{4}\dot{\boldsymbol{\rho}}^2(\tau)+U(|\boldsymbol{\rho}(\tau)|)\right]}$$

$$= \sum_n \Psi_n(\mathbf{x}) e^{-E_n X}\Psi_n^+(\mathbf{x}').$$

The Green function satisfies the correlation for $t > t'' > t'$

$$G_X(\mathbf{x},\mathbf{x}') = \int d\mathbf{y}\ G_{X-t''}(\mathbf{x},\mathbf{y})G_{t''}(\mathbf{y},\mathbf{x}')$$

Average

$$I = \left\langle e^{i\frac{\mathbf{k}}{2}(\boldsymbol{\rho}(\tau_1)+\boldsymbol{\rho}(\tau_2))} \right\rangle = \int d\sigma \;\; e^{i\frac{\mathbf{k}}{2}(\boldsymbol{\rho}(\tau_1)+\boldsymbol{\rho}(\tau_2))}$$

$$= \int d\boldsymbol{\rho}_1 \int d\boldsymbol{\rho}_2 \;\; G_{X-\tau_1}(0,\boldsymbol{\rho}_1) e^{i\frac{\mathbf{k}}{2}\boldsymbol{\rho}_1} G_{\tau_1-\tau_2}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) e^{i\frac{\mathbf{k}}{2}\boldsymbol{\rho}_2} G_{\tau_2}(\boldsymbol{\rho}_2, 0)$$

$$= \sum_{n_1 n_2 n_3} \iint d\boldsymbol{\rho}_1 d\boldsymbol{\rho}_2 \;\; e^{-E_{n_1}(X-\tau_1)-E_{n_2}(\tau_1-\tau_2)-E_{n_3}\tau_2} \Psi_{n_1}(0)\Psi^*_{n_3}(0)$$

$$\cdot \Psi^*_{n_1}(\boldsymbol{\rho}_1) e^{i\frac{\mathbf{k}}{2}\boldsymbol{\rho}_1} \Psi_{n_2}(\boldsymbol{\rho}_1) \Psi^*_{n_2}(\boldsymbol{\rho}_2) e^{i\frac{\mathbf{k}}{2}\boldsymbol{\rho}_2} \Psi_{n_3}(\boldsymbol{\rho}_2) \qquad (n_1 = n_3)$$

$$\Rightarrow e^{-E_0 X} |\Psi_0(0)|^2 \sum_n e^{-(E_n - E_0)|\tau_1-\tau_2|}$$

$$\cdot \left(\int d\boldsymbol{\rho}_1 \Psi^*_0(\boldsymbol{\rho}_1) e^{i\frac{\mathbf{k}}{2}\boldsymbol{\rho}_1} \Psi_n(\boldsymbol{\rho}_1) \right) \left(\int d\boldsymbol{\rho}_2 \Psi^*_n(\boldsymbol{\rho}_2) e^{i\frac{\mathbf{k}}{2}\boldsymbol{\rho}_2} \Psi_0(\boldsymbol{\rho}_2) \right)$$

$$= e^{-E_0 X} |\Psi_0(0)|^2 \sum_n e^{-(E_n - E_0)|\tau_1-\tau_2|} C_{0n}\left(\frac{\mathbf{k}}{2}\right) C^*_{0n}\left(-\frac{\mathbf{k}}{2}\right)$$

"Time"potential

$$H = P^2 + \frac{\alpha}{8}|\rho|; \qquad \left[-\frac{d^2}{d\rho^2} + \frac{\alpha}{8}|\rho|\right]\Phi_n(\rho) = \mathcal{E}_n\Phi_n(\rho)$$

$$\rho = \frac{2v}{\alpha^{\frac{1}{3}}}, \qquad \mathcal{E} = \frac{\alpha^{\frac{2}{3}}}{4}\epsilon$$

$$\left[-\frac{d^2}{dv^2} + v\right]Y(v,\epsilon) = \epsilon Y(v,\epsilon), \qquad v \in [0,\infty).$$

$$Y(v,\epsilon) = \begin{cases} \pi\sqrt{\frac{\varepsilon - v}{3}}\left[J_{\frac{1}{3}}\left(\frac{2}{3}(\varepsilon - v)^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}(\varepsilon - v)^{\frac{3}{2}}\right)\right], & v < \varepsilon \\ \sqrt{v - \varepsilon}K_{\frac{1}{3}}\left(\frac{2}{3}(v - \varepsilon)^{\frac{3}{2}}\right), & v > \varepsilon \end{cases}$$

The spectrum is defined by the equations

$$\frac{d}{dv} Y(v, \epsilon_{2n})\bigg|_{v=0} = 0 \qquad \text{for even states} \qquad n \to 2n$$
$$Y(0, \epsilon_{2n+1}) = 0 \qquad \text{for odd states} \qquad n \to 2n+1.$$

$$\Phi_n(v) = Y(v, \epsilon_n)$$

n	ϵ_n	$N_n = \int_0^\infty dv \ \Phi_n^2(v)$
0	1.0188	8.655
1	2.3381	14.558
2	3.2482	16.886
3	4.0879	19.097
4	4.8201	20.652