# Determination of the mass spectrum of $(c\bar{b})$ mesons with relativistic corrections

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# Plan

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## 1. The problems of bound state system

Currently there exist the following conception of the structure and mechanism of formation of quantum objects: bound states are composed of fermions and the interactions between the fermions are carried out through the exchange of bosons. Particularly, in an atomic structure consisting of electrons and nuclei such bosons are photons, nucleons in a nucleus are bound by mesons, while hadrons consisting of quarks by gluons.

- However, in modern relativistic quantum field theory (QFT) formation and description of bound states is still not a well-posed problem. It describes the elastic and inelastic scattering of free relativistic particles located at large distances from each other and constituting plane waves.
- On the other hand, it is well known that the energy spectrum of a bound state can be determined accurately within the non-relativistic quantum mechanics (NRQM) with an appropriate choice of the interaction potential.
- Thus, real physics requires creation of a mathematical theory that describes bound states on the basis of QFT. All the efforts spent in this direction can be divided into two categories.

- The starting point in one direction is the assertion that if there are bound states of two particles with the appropriate quantum numbers, then the amplitude of elastic scattering of these particles has a simple pole in the energy at the mass of the bound state. The Bethe-Salpeter equation and the so-called quasipotential equation was formulated on the basis of this idea.

- Another trend is based on the belief that the non-relativistic SE is a reliable tool for investigating and determining the energy spectrum of bound states. Meanwhile the true relativistic corrections are small, so that the theoretical problem is reduced to obtaining relativistic corrections to nonrelativistic interaction potential based on the formalism of QFT.

- There is another approach based on the following idea. Exact solutions for quantum field Green's functions can be formally represented in the form of functional integrals. Computing functional integrals is still in its immaturity. However, the available representations can be used to obtain solutions to the onrelativistic Schredinger equation in the form of Feynman functional integral with a potential containing the necessary relativistic corrections. Not much work have been done in this direction. Our studies continue these efforts.

## 2. The mass spectrum bound state system

- In this section, we will present one of the alternative methods of the bound state mass determination when the nonperturbative and relativistic character of the interaction is taken into account. The polarization loop function for a scalar particle can be written as

$$\begin{aligned}\Pi(x-y) &= \langle J(x)J(y)\rangle = \langle \Phi^{+}(x)\Phi(x)\Phi^{+}(y)\Phi(y)\rangle \\ &= \langle G_{m_1}(x,y|A)G_{m_2}(y,x|A)\rangle_A \ . \end{aligned} \quad (2.1)$$

Here the averaging over the external gauge field $A_\alpha(x)$ is performed. The Green function $G_m(y,x|A)$ for the scalar particle in the external gauge field is determined from the equation

$$\left[\left(i\frac{\partial}{\partial x_\alpha}+\frac{g}{c\hbar}\cdot A_\alpha(x)\right)^2+\frac{c^2m^2}{\hbar^2}\right]G_m(x,y|A)=\delta(x-y)\ , \tag{2.2}$$

where *m* is the mass of the scalar particle, and *g* is the coupling constant. In averaging over the external gauge field $A_\alpha(x)$ ; let us consider only the lowest order or only the two-point Gauss correlator

$$<\exp\left\{i\int dx A_\alpha(x)J_\alpha(x)\right\}>_A=\exp\left\{-\frac{1}{2}\int\int dxdy J_\alpha(x)D_{\alpha\beta}(x-y)J_\beta(y)\right\} \tag{2.3}$$

where $J_\alpha(x)$ is the real current. The propagator of the gauge field has the following form:

$$D_{\alpha\beta}(x-y) =< A_\alpha(x)A_\beta(y) >_A \qquad (2.4)$$

The mass of the bound state is usually defined through the loop function in the following way:

$$M = -\lim_{|x-y|\to\infty} \frac{\ln \Pi(x-y)}{|x-y|} \, . \qquad (2.5)$$

Thus, if we know the loop function, then we can determine the bound state mass. The solution of (2.2) can be represented as a functional integral in the following way ( Dineykhan M., Efimov G.V. and Namsrai Kh. Fortschr. Phys. 1991.V.39. P. 259. )

$$G_m(x,y|A) = \int_0^\infty \frac{ds}{(4s\pi)^2} \exp\left\{-sm^2 - \frac{(x-y)^2}{4s}\right\} \int d\sigma_\beta \exp\left\{ig \int_0^1 d\xi \frac{\partial Z_\alpha(\xi)}{\partial \xi} A_\alpha(\xi)\right\}$$

(2.6)

where the following notation is used:

$$Z_\alpha(\xi) = (x-y)_\alpha \xi + y_\alpha - 2\sqrt{s} B_\alpha(\xi);$$

$$d\sigma_\beta = N \delta B_\beta \exp\left\{ -\frac{1}{2} \int_0^1 d\xi B'^2(\xi) \right\}, \tag{2.7}$$

with the normalization

$B_\alpha(0) = B_\alpha(1) = 0$ ; and $\int d\sigma_\beta = 1,$

where *N* is the normalization constant. Substituting (2.6) into (2.1) and performing averaging over the external gauge field one can obtain for the loop function

$$\Pi(x) = \iint_0^\infty \frac{d\mu_1 d\mu_2}{(8\pi^2 x)^2} J(\mu_1, \mu_2) \exp\left\{ -\frac{|x|}{2}\left(\frac{m_1^2}{\mu_1} + \mu_1\right) - \frac{|x|}{2}\left(\frac{m_2^2}{\mu_2} + \mu_2\right) \right\}. \tag{2.8}$$

Here

$$J(\mu_1,\mu_2) \;=\; N_1N_2 \iint \delta r_1 \delta r_2 \exp\left\{ -\frac{1}{2}\int_0^x d\tau \left[\mu_1 \dot{r}_1^2(\tau) + \mu_2 \dot{r}_2^2(\tau)\right]\right\} e^{-W} \;; \tag{2.9}$$

$$W = W_{1,1} + W_{2,2} - 2W_{1,2} \;,$$

and the following notation is introduced:

$$W_{i,j} = \frac{g^2}{2}(-1)^{i+j}\int_0^x\int_0^x d\tau_1 d\tau_2 Z'^{(i)}{}_{\alpha}(\tau_1) D_{\alpha\beta}\left(Z^{(i)}(\tau_1) - Z^{(j)}(\tau_2)\right) Z'^{(j)}{}_{\beta}(\tau_2) \;. \tag{2.10}$$

We determine the polarization loop function for two charged scalar particles in the external gauge field with masses $m_1,\, m_2$. On the other hand, the functional integral represented in (2.9) is analogous to the Feynman path integral for the motion of two particles with masses $\mu_1,\, \mu_2$ in the non-relativistic quantum mechanics. The interaction between these particles is described by expression (2.10) which contains the potential and nonpotential parts, in particular, $W_{1,1},\, W_{2,2}$ define nonpotential interactions, and $W_{1,2}$ ; defines potential interactions of a nonlocal nature.

In the asymtotic $|x| \to \infty$ the integral (2.9) represented as:

$$\lim_{|x|\to\infty} J(\mu_1, \mu_2) \Longrightarrow \exp\{-x \cdot E(\mu_1, \mu_2)\} \; . \qquad (2.11)$$

According to (2.5), one needs to derive the loop function in asymptotics. In this approximation the integral in (2.9) is evaluated by the saddle-point technique and, hence, for the bound state mass we obtain

$$M = \frac{1}{2} \min_{\mu_1, \mu_2} \left\{ \frac{m_1^2}{\mu_1} + \mu_1 + \frac{m_2^2}{\mu_2} + \mu_2 + 2E(\mu_1, \mu_2) \right\} \; . \qquad (2.12)$$

Then the interaction Hamiltonian can be represented in the form

$$H = \frac{1}{2\mu_1} \mathrm{P}_1^2 + \frac{1}{2\mu_2} \mathrm{P}_2^2 + V(\mathbf{r}_1, \mathbf{r}_2) \qquad (2.13)$$

From the SE

$$H\Psi(\mathbf{r}_1, \mathbf{r}_2) = E(\mu_1, \mu_2)\Psi(\mathbf{r}_1, \mathbf{r}_2) \ , \qquad (2.14)$$

the $E(\mu)$ - eigenvalues of the Hamiltonian (2.13) can be determined. In the this case $\mu_j$:

$$\mu_j - \frac{m_j^2}{\mu_j} + 2\mu_j \frac{dE(\mu_1, \mu_2)}{d\mu_j} = 0 \ ; \quad j = 1, \ 2 \ . \qquad (2.15)$$

the parameters we considered as $\mu_1, \mu_2$ mass.

$$\frac{1}{\mu} = \frac{1}{\mu_1} + \frac{1}{\mu_2} \qquad (2.16)$$

$$M = \mu_1 + \mu_2 + \mu\frac{dE}{d\mu} + E(\mu)$$

$$E(\mu_1, \mu_2) = E(\mu) \qquad (2.17)$$

$$\mu_1 = \sqrt{m_1^2 - 2\mu^2\frac{dE}{d\mu}} \ ; \quad \mu_2 = \sqrt{m_2^2 - 2\mu^2\frac{dE}{d\mu}} \qquad (2.18)$$

# 3. The nonperturbative correction for the interaction hamiltonian

In the standard evaluation, when the relativistic character of interaction is taken into account, one usually restricts oneself to the lower order of the $(v/c)$ value, but in our case the ultra-relativistic limit is also included, namely, we determine the interaction type by summation of the infinite series in power of . N$(v/c)$we start the determination of the interaction Hamiltonian structure. Taking (2.10) into account and using the Fourier transformation of the gluon propagator let us rewrite the expression for in $W_{i,j}$ following way:

$$
\begin{aligned}
W_{i,j} &= \frac{g^2}{2}(-1)^{i+j} \cdot \int\limits_0^x \int\limits_0^x d\tau_1 d\tau_2 \cdot \left(\vec{n} + \frac{1}{c}\vec{r}_i'(\tau_1)\right) \cdot \left(\vec{n} + \frac{1}{c}\vec{r}_j'(\tau_2)\right) \\
&\times \int \frac{d\vec{q}}{(2\pi)^3} \int\limits_{-\infty}^{\infty} \frac{ds}{2\pi} \widehat{D}\left(\vec{q}^2 + \frac{s^2}{c^2}\right) \\
&\times \exp\left\{ is(\tau_1 - \tau_2) + \frac{is}{c}\left(r_i^{(4)}(\tau_1) - r_j^{(4)}(\tau_2)\right) + i\vec{q}(\vec{r}_i(\tau_1) - \vec{r}_j(\tau_2)) \right\}
\end{aligned} \qquad (3.19)
$$

where $\vec{n} = \vec{r}/r$, $\vec{r} = \vec{r}_1(\tau) - \vec{r}_2(\tau)$, $r = |\vec{r}|$ and is the Fouri $\widehat{D}$ image of the $D$ function. The interaction between constituent particles is caused by the interchange of the gauge field quanta so let us write the propagator in the standard way:

$$\widehat{D}\left(\vec{q}^2 + \frac{s^2}{c^2}\right) \simeq \frac{1}{\vec{q}^2 + \frac{s^2}{c^2}} = \int_0^\infty d\eta \cdot \exp\left\{-\eta \cdot \left(\vec{q}^2 + \frac{s^2}{c^2}\right)\right\} \tag{3.20}$$

According to (3.19), after integration over $d\vec{q}$ we have for the interaction potential

$$\begin{aligned} W_{i,j} &= \frac{2}{3} g^2 (-1)^{i+j} \int_0^t \int_0^t d\tau_1 d\tau_2 \int_{-\infty}^{\infty} \frac{ds}{2\pi} \int_0^\infty \frac{d\eta}{(2\sqrt{\pi\eta})^3} \\ &\times \exp\left\{-\frac{\vec{r}^2}{4\eta}\right\} \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{(-1)^{n+k}}{n!(k-n)!} \eta^n r^{(4)(k-n)} \left(\frac{is}{c}\right)^{n+k} e^{is\tau} \Theta_{ij} \ , \end{aligned} \tag{3.21}$$

where the following notation is introduced:

$$\tau = (\tau_1 - \tau_2)\ ;$$

$$r^{(4)} = r_i^{(4)}(\tau_1) - r_j^{(4)}(\tau_2)\ ; \qquad (3.22)$$

$$\Theta_{ij} = 1 + \frac{\vec{n}}{c} \cdot \left(\vec{r}_i{}'(\tau_1) + + \vec{r}_j{}'(\tau_2)\right) + \frac{\vec{r}_i{}'(\tau_1)\vec{r}_j{}'(\tau_2)}{c^2}$$

Here $\tau_1$ and $\tau_2$ are considered as the proper time of the constituent particles 1 and 2,

$$r^{(4)} = c(\tau_1 - \tau_2)u \equiv c\tau u \qquad (3.23)$$

where *u* is a new variable. Taking into account (3.22) and (3.23) and integrating over $ds$ and, $du$ after some simplifications, from (3.21) we obtain

$$\begin{aligned} W_{i,j} &= (-1)^{i+j} \cdot \frac{g^2}{6\pi} \cdot \int_0^t \int_0^t d\tau_1 d\tau_2 \frac{\delta(\tau_1 - \tau_2)}{|\vec{r}_j(\tau_1) - \vec{r}_j(\tau_2)|} \\ &+ (-1)^{i+j} \cdot \frac{g^2}{6\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!c^{2k}} \cdot \int_0^t d\tau \frac{\partial^{2k}}{\partial \tau^{2k}} \\ &\times \left(|\vec{r}_i(\tau) - \vec{r}_j(\tau)|\right)^{2k-1} \equiv W_{i,j}^{(1)} + W_{i,j}^{(2)} \; . \end{aligned} \qquad (3.24)$$

Let us consider each term in (3.24) separately. The first term corresponds to the contribution of the one-photon (one-gluon) exchange and contains the diagonal and nondiagonal interaction

$$-W_{1,1}^{(1)} + 2W_{1,2}^{(1)} - W_{2,2}^{(1)} = \int_0^t d\tau \left\{ -\frac{4}{3}\frac{\alpha_s}{r(\tau)} + V(0) \right\} \quad (3.25)$$

where

$$r(\tau) = |\vec{r}_1(\tau) - \vec{r}_2(\tau)|;$$

$$V(0) = \frac{4}{3}\alpha_s \int \frac{d\vec{q}}{(2\pi)^2} \cdot \frac{1}{\vec{q}^2} \quad (3.26)$$

In this case, from (3.24) and (3.25) we can see that the diagonal interaction, when $i = j$, defines the mass renormalization. So *V* (0) conforms with the ordinary mass operator renormalization in the nonrelativistic limit and will be considered as a constant parameter in the following calculations. Simonov Yu.A. Phys. Lett.B. 2001 .V.515. P.137.

The second term $W_{i,j}^{(2)}$ in Eq.(3.24) contains only the cross items $(i \neq j)$. The diagonal terms are equal to zero. Combined result for the interaction Hamiltonian with the nonperturbative correction coming from (3.25) reads

$$H = H^0 + \triangle H^0_{nonper} \tag{3.27}$$

where $H^0$ is the nonrelativistic Hamiltonian and $\triangle H^0_{nonper}$ is the nonperturbative correction

$$H^0 = \frac{1}{2\mu} \cdot \vec{P}^2 - \frac{4}{3}\frac{\alpha_s}{r} + V(0)\ ;$$
$$\triangle H^0_{nonper} = -\frac{\alpha_s}{r} \cdot \left[ \frac{1}{\sqrt{1 + \ell(\ell+1)/(4\hbar^2 c^2 r^2 \mu^2)}} - 1 \right] \tag{3.28}$$

Thus, we have obtained the nonperturbative correction to the interaction Hamiltonian which is related to the relativistic nature of the system.. Dineykhan M., Zhaugasheva S.A., // Few-Body Systems. 2005. V.37. P.49-69.

# 4. The interaction Hamiltonian

The total Hamiltonian:

$$H = H_c + H_{spin} + \triangle H_{np} \tag{4.1}$$

where

$$H_c = \frac{1}{2\mu}\mathbf{P}^2 + \sigma\ r - \frac{4}{3}\frac{\alpha_s}{r} \tag{4.2}$$

is central hamiltonian

$$H_{spin} = H_{SS} + H_{LS} + H_{TT} \tag{4.3}$$

is the spin-orbit interaction

$$H_{SS} = \frac{2}{3\mu_1\ \mu_2}\ (\mathbf{S_1 S_2})\ \Delta V_{\mathrm{v}} \tag{4.4}$$

is spin-spin

$$H_{LS} = \frac{1}{4\,\mu_1^2\mu_2^2}\frac{1}{r}\left\{\left[\left((\mu_1+\mu_2)^2+2\mu_1\mu_2\right)(\mathbf{L}\cdot\mathbf{S}_+)+(\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)\right]\frac{\partial}{\partial r}V_{\mathrm{v}} - \left[(\mu_1^2+\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_+)+(\mu_1^2-\mu_2^2)(\mathbf{L}\cdot\mathbf{S}_-)\right]\frac{\partial}{\partial r}V_s\right\}, \quad (4.5)$$

is spin-orbit

$$H_{TT} = \frac{1}{12\mu^2}\left[\frac{1}{r}\frac{\partial}{\partial r}V_{\mathrm{v}} - \frac{\partial^2}{\partial r^2}V_{\mathrm{v}}\right] S_{12} \quad (4.6)$$

is tensor. The potentials:

$$V_{\mathrm{v}} = -\frac{4\alpha_s}{3}\frac{1}{r}\ ; \qquad V_s = r\sigma\ ; \quad (4.7)$$

Used the notations

$$\mathbf{S}_+ = \mathbf{S_1}+\mathbf{S_2}\ ; \quad \mathbf{S}_- = \mathbf{S_1}-\mathbf{S_2}\ ;$$

$$S_{12} = \frac{4}{(2\ell+3)(2\ell-1)}\left[\mathbf{L}^2\mathbf{S}^2 - \frac{3}{2}(\mathbf{LS}) - 3(\mathbf{LS})^2\right] \quad (4.8)$$

The nonperturbative Hamiltonian

$$\triangle H_{np} = -\frac{4}{3}\frac{\alpha_s}{r}\left[\frac{1}{\sqrt{1+\frac{\ell(\ell+1)}{4\mu^2 r^2}}} - 1\right] \qquad (4.9)$$

## 5. The energy spectrum total amiltonian with relativistic corrections.

We consider the SE:

$$[H_c + H_{spin} + \triangle H_{np}]\,\Psi = E\Psi \qquad (5.1)$$

The energy spectrum and wave function are determined from SE in the framework of the oscillator representation (OR) method and let us change the variables in the following way

$$r = q^{2\rho}\,, \qquad \Psi \Rightarrow \Psi(q^2) = q^{2\rho\ell} \cdot \Phi(q^2) \qquad (5.2)$$

Here $\rho$ *is the variational parameter connected with asymptotic behavior of the wave function.* SE in $R^d$

$$\Big\{ -\frac{1}{2}\left(\frac{\partial^2}{\partial q^2}+\frac{d-1}{q}\frac{\partial}{\partial q}\right)-4\rho^2\,\mu E\,q^{2(2\rho-1)}+4\rho^2\,\mu\,\sigma\,q^{2(3\rho-1)}-$$

$$-\frac{16\,\rho^2\,\mu\,\alpha_s}{3}\cdot\frac{q^{2(\rho-1)}}{\sqrt{1+\ell(\ell+1)/(4\mu^2q^{4\rho})}}+\frac{64\alpha_s\mu\rho^2}{9\mu_1\mu_2}\cdot\frac{\ell}{q^{2(\rho+1)}}\cdot(\vec{S}_1\vec{S}_2)-$$

$$-\left(\frac{1}{\mu_1^2}+\frac{1}{\mu_2^2}\right)\cdot\frac{\sigma\mu\rho^2}{3}\,(\vec{L}\vec{S})\,q^{2(\rho-1)}+\frac{4\mu\rho^2\alpha_s}{3q^{2(\rho+1)}}\left[\frac{S_{12}}{\mu_1\mu_2}+\frac{(\vec{L}\vec{S})}{3}\left(\frac{1}{\mu^2}+\frac{2}{\mu_1\mu_2}\right)\right]\Big\}\Phi(q^2)=0 \qquad (5.3)$$

where *d is the dimension of* $R^d$

$$d=2+2\rho+4\rho\ell \qquad (5.4)$$

We consider the bound state systems whit $(c\,\bar{c})$ *and* $(b\,\bar{b})$ *quarks. In the this case the mass* and constituent mass determined from

$$4\mu^2=m_q^2-2\mu^2\frac{dE}{d\mu} \qquad (5.5)$$

Introduce the new variables

$$\omega^{\rho} = Z \cdot \sqrt{\sigma}\ ; \quad \mu = x \cdot \sqrt{\sigma}\ ; \quad x = Z \cdot u \qquad (5.6)$$

The energy spectrum singlet state:

$$\begin{aligned}\frac{E_s}{\sqrt{\sigma}} &= \min_{\rho}\left\{\frac{Z_s^2\Gamma(2+\rho+2\rho\ell)}{8\rho^2 x_s\Gamma(3\rho+2\rho\ell)} + \frac{1}{Z_s}\frac{\Gamma(4\rho+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)} - \frac{4\alpha_s Z_s}{3}\frac{\Gamma(2\rho+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)} - \right. \\ &- \left. \frac{\alpha_s Z_s^3}{6x_s^2\rho}\frac{\Gamma(1+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)} + \frac{\alpha_s Z_s^3(1+\ell)}{12x_s^2\rho}\frac{\Gamma(1+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)}\right\}\ ,\end{aligned} \qquad (5.7)$$

and for the triplet:

$$\begin{aligned}\frac{E_t}{\sqrt{\sigma}} &= \min_{\rho}\left\{\frac{Z_t^2\Gamma(2+\rho+2\rho\ell)}{8\rho^2 x_t\Gamma(3\rho+2\rho\ell)} + \frac{1}{Z_t}\frac{\Gamma(4\rho+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)} - \frac{4\alpha_s Z_t}{3}\frac{\Gamma(2\rho+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)} + \right. \\ &+ \frac{\alpha_s Z_t^3}{18x_t^2\rho}\frac{\Gamma(1+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)} + \frac{\alpha_s Z_t^3(1+\ell)}{12x_t^2\rho}\frac{\Gamma(1+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)} - \frac{Z_t\ \ell}{24x_t^2}\frac{\Gamma(2\rho+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)} + \\ &+ \left. \frac{\alpha_s\ Z_t^3\ (1+\ell)}{6\ x_t^2\ \rho\ (2+3\ell)} \cdot \frac{\Gamma(1+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)}\right\}\ .\end{aligned} \qquad (5.8)$$

From (5.5) we obtained the equation for *u. For* $S=0$*:*

$$u_s^3-\frac{u_s}{16\rho^2}\cdot\frac{\Gamma(2+\rho+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)}+\frac{\alpha_s}{6\rho}\cdot\frac{\Gamma(1+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)}-\frac{\alpha_s(1+\ell)}{12\rho}\cdot\frac{\Gamma(1+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)}-\frac{m_q^2}{4\,\sigma}\frac{u_s}{Z_s^2}=0 \tag{5.9}$$

and for *S = 1:*

$$u_t^3-\frac{u_t}{16\rho^2}\cdot\frac{\Gamma(2+\rho+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)}-\frac{\alpha_s}{18\rho}\cdot\frac{\Gamma(1+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)}-\frac{\alpha_s(1+\ell)}{12\rho}\cdot\frac{\Gamma(1+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)}-$$
$$-\frac{\alpha_s}{6\rho}\frac{(1+\ell)}{(3+2\ell)}\cdot\frac{\Gamma(1+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)}-\frac{m_q^2}{4\,\sigma}\frac{u_s}{Z_s^2}=0\ . \tag{5.10}$$

The parameters $Z_s$ *and* $Z_t$ *is determined as*

$$Z_s^2=\frac{4\rho^2\Gamma(4\rho+2\rho\ell)u_s}{W_s}\ ;\qquad Z_t^2=\frac{4\rho^2\Gamma(4\rho+2\rho\ell)u_t+\rho^2\ell\Gamma(2\rho+2\rho\ell)/(6u_t)}{W_t} \tag{5.11}$$

where used the notations:

$$W_s = \Gamma(2+\rho+2\rho\ell) - \frac{16}{3}\alpha_s\, u_s\rho^2\Gamma(2\rho+2\rho\ell) - \\ - \frac{2\alpha_s\rho}{u_s}\Gamma(1+2\rho\ell) + \frac{\alpha_s\rho\,(1+\ell)}{u_s}\Gamma(1+2\rho\ell)\;; \qquad (5.11)$$

$$W_t = \Gamma(2+\rho+2\rho\ell) - \frac{16}{3}\alpha_s\, u_t\rho^2\Gamma(2\rho+2\rho\ell) + \frac{2\alpha_s\rho}{3u_t}\Gamma(1+2\rho\ell) + \\ + \frac{\alpha_s\rho\,(1+\ell)}{u_t}\Gamma(1+2\rho\ell) + \frac{\alpha_s\rho}{u_t}\frac{(1+\ell)}{(3+2\ell)}\Gamma(1+2\rho\ell)\;; \qquad (5.12)$$


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**Table 1.** Mass spectrum $(c\bar{c})$ system in $GeV$.
With $\alpha_s = 0.2$, $\sigma = 0.26\ GeV^2$.

| | $\ell$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $S = 0$ | $\rho_s$ | 0.764 | .608 | .585 | .585 |
| | $u_s$ | .7624 | 1.196 | 1.3486 | 1.416 |
| | $E_s$ | .6342 | 1.0764 | 1.3999 | 1.6759 |
| | $x_s$ | 1.4727 | 1.5318 | 1.5851 | 1.6346 |
| | $z_s$ | 1.9317 | 1.2808 | 1.1753 | 1.1543 |
| | $\mu_{qs}$ | 1.5018 | 1.5621 | 1.6164 | 1.661 |
| | $M_{sp}$ | 2.983 | 3.413 | 3.8035 | 4.106 |
| $S = 1$ | $\rho_t$ | .554 | .574 | .576 | .576 |
| | $u_t$ | 1.2522 | 1.321 | 1.3917 | 1.4549 |
| | $E_t$ | .6893 | 1.0899 | 1.4042 | 1.6706 |
| | $x_t$ | 1.4946 | 1.5358 | 1.583 | 1.6291 |
| | $z_t$ | 1.1935 | 1.1626 | 1.1375 | 1.1197 |
| | $\mu_{qt}$ | 1.5242 | 1.5662 | 1.6143 | 1.6613 |
| | $M_{pt}$ | 3.089 | 3.508 | 3.8067 | 4.1033 |

**Table 2.** Mass spectrum $(b\bar{b})$ in $GeV$.
With $\alpha_s = 0.16$, $\sigma = 0.26\ GeV^2$.

| | $\ell$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| $S = 0$ | $\rho_s$ | 0.74 | .623 | .6 | .59 |
| | $u_s$ | 1.6305 | 2.5738 | 2.6275 | 2.7946 |
| | $E_s$ | .3457 | .6916 | .9365 | 1.1450 |
| | $x_s$ | 4.6468 | 4.6681 | 4.6983 | 4.728 |
| | $z_s$ | 2.85 | 1.9802 | 1.7881 | 1.6918 |
| | $\mu_{qs}$ | 4.7389 | 4.7605 | 4.7914 | 4.8211 |
| | $M_{sp}$ | 9.445 | 9.7927 | 10.1405 | 10.2522 |
| $S = 1$ | $\rho_t$ | .631 | .61 | .593 | .575 |
| | $u_t$ | 2.1053 | 2.4543 | 2.6871 | 2.8209 |
| | $E_t$ | .3682 | .69695 | .9388 | 1.1459 |
| | $x_t$ | 4.6512 | 4.6698 | 4.6987 | 4.7277 |
| | $z_t$ | 2.2093 | 1.9027 | 1.7487 | 1.6759 |
| | $\mu_{qt}$ | 4.7434 | 4.7623 | 4.7918 | 4.8213 |
| | $M_{pt}$ | 9.4678 | 9.873 | 10.043 | 10.253 |

**Table 3.** Mass spectrum of charmed mesons in GeV.

| | $\ell$ | [1] | [2] | [3] | *Exp* [4] | *Our* |
|---|---|---|---|---|---|---|
| $S = 0$ | 0 | 2.979 | 2.979 | 2.9879 | $2.9798 \pm .0018$ | 2.983 |
| | 1 | 3.415 | 3.424 | 3.415 | $3.4150 \pm .0008$ | 3.413 |
| | 2 | | 3.811 | | $3.840 \pm .02$ | 3.8035 |
| | 3 | | | | 4.090 | 4.106 |
| $S = 1$ | 0 | 3.097 | 3.096 | 3.0969 | $3.09687 \pm .00004$ | 3.089 |
| | 1 | 3.511 | 3.510 | 3.51113 | $3.5105 \pm 0.00012$ | 3.508 |
| | 2 | | 3.815 | 3.819 | | 3.8067 |
| | 3 | | | | 4.090 | 4.1033 |

**Table 4.** Mass spectrum of bottom mesons in GeV.

| | $\ell$ | [1] | [2] | [3] | *Exp*[4] | *Our* |
|---|---|---|---|---|---|---|
| $S = 0$ | 0 | 9.402 | 9.400 | 9.4076 | $9.300 \pm .02$ | 9.445 |
| | 1 | 9.847 | 9.863 | 9.8619 | $9.8598 \pm 0.0013$ | 9.7927 |
| | 2 | | 10.153 | | 10.161 | 10.1405 |
| | 3 | | | | | 10.2522 |
| $S = 1$ | 0 | 9.465 | 9.460 | 9.4603 | $9.4603 \pm .00021$ | 9.4678 |
| | 1 | 9.876 | 9.892 | 9.8934 | $9.8919 \pm 0.7$ | 9.873 |
| | 2 | 10.138 | 10.158 | | | 10.043 |
| | 3 | | | | 10.2325 | 10.253 |

# 6.The mass spectrum ($c\bar{b}$) *mesons*

The quark mass:

$$m_c = 1.27 + 0.07 - 0.11\ GeV\ ; \qquad m_b = 4.50 + 0.17 - 0.07\ Gev.$$

In the this case $\mu_1 \neq \mu_2$

$$\sin^2(2\varphi) = \frac{4\mu_1\mu_2}{(\mu_1 + \mu_2)^2} \tag{6.1}$$

For the energy spectrum with *S = 0:*

$$\begin{aligned} \frac{E_s}{\sqrt{\sigma}} &= \min_{\rho}\left\{\frac{Z_s^2\Gamma(2+\rho+2\rho\ell)}{8\rho^2 x_s\Gamma(3\rho+2\rho\ell)} + \frac{1}{Z_s}\frac{\Gamma(4\rho+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)} - \frac{4\alpha_s Z_s}{3}\frac{\Gamma(2\rho+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)} - \right. \\ &\quad - \left. \frac{\alpha_s Z_s^3}{6x_s^2\rho}\frac{\Gamma(1+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)}\sin^2(2\varphi_s) + \frac{\alpha_s Z_s^3(1+\ell)}{12x_s^2\rho}\frac{\Gamma(1+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)}\right\}, \end{aligned} \tag{6.2}$$

and for *S = 1:*

$$\frac{E_t}{\sqrt{\sigma}} = \min_{\rho}\left\{\frac{Z_t^2\Gamma(2+\rho+2\rho\ell)}{8\rho^2 x_t\Gamma(3\rho+2\rho\ell)} + \frac{1}{Z_t}\frac{\Gamma(4\rho+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)} - \frac{4\alpha_s Z_t}{3}\frac{\Gamma(2\rho+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)} + \right.$$
$$+ \frac{\alpha_s Z_t^3}{18x_t^2\rho}\frac{\Gamma(1+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)}\sin^2(2\varphi_t) + \frac{\alpha_s Z_t^3(1+\ell)}{12x_t^2\rho}\frac{\Gamma(1+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)} - \frac{Z_t\,\ell}{24x_t^2}\frac{\Gamma(2\rho+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)}\times \quad (6.3)$$
$$\left.\times\left(1-\frac{1}{2}\sin^2(2\varphi_t)\right) + \frac{\alpha_s\,Z_t^3}{18x_t^2\,\rho}\left(\frac{\ell}{(2+3\ell)}\sin^2(2\varphi_t)+1\right)\cdot\frac{\Gamma(1+2\rho\ell)}{\Gamma(3\rho+2\rho\ell)}\right\}.$$

**Table 5.** The values of $\sin^2(2\varphi)$

| | $\ell$ | 0 | 1 | 2 |
|---|---|---|---|---|
| $(u\bar{s})$ | $s=0$ | 0.992874 | 0.99967 | 0.99977 |
| | $s=1$ | 0.99776 | 0.99969 | 0.99974 |
| $(u\bar{c})$ | $s=0$ | 0.74577 | 0.7953 | 0.81327 |
| | $s=1$ | 0.85899 | 0.80846 | 0.80879 |
| $(u\bar{b})$ | $s=0$ | 0.3425 | 0.39544 | 0.41627 |
| | $s=1$ | 0.3944 | 0.40395 | 0.4118 |
| $(s\bar{c})$ | $s=0$ | 0.80775 | 0.80833 | 0.82376 |
| | $s=1$ | 0.808376 | 0.82065 | 0.820992 |
| $(s\bar{b})$ | $s=0$ | 0.3919 | 0.40314 | 0.4259 |
| | $s=1$ | 0.40764 | 0.414885 | 0.4265 |
| $(c\bar{b})$ | $s=0$ | 0.73095 | 0.7441 | 0.7545 |
| | $s=1$ | 0.73619 | 0.74495 | 0.75398 |

**Table 6.** The energy and mass spectrum $(c\bar{b})$ mesons and constituents masses $c$ and $b$ quarks, in GeV.

| | $\ell$ | 0 | 1 | 2 |
|---|---|---|---|---|
| $S = 0$ | $E_s$ | 0.7019 | 1.1204 | 1.4338 |
| | $\mu_1$ | 1.4914 | 1.551 | 1.6073 |
| | $\mu_2$ | 4.6492 | 4.669 | 4.6877 |
| | $M_s$ | 6.2873 | 6.690 | 6.9932 |
| | [2] | 6.270 | 6.699 | 7.072 |
| $Exp$ | Exp [4] | $6.276 \pm 0.004$ | $6.720 \pm 0.0027$ | |
| $S = 1$ | $E_t$ | 0.7359 | 1.1301 | 1.4373 |
| | $\mu_1$ | 1.5066 | 1.5608 | 1.6153 |
| | $\mu_2$ | 4.6541 | 4.672 | 4.6905 |
| | $M_t$ | 6.3165 | 6.70 | 6.9958 |
| | [2] | 6.332 | 6.734 | 7.077 |

# 7. The Radiative and Leptonic decay of mesons

We consider the *E1 transition and the width determined in the form*

$$\Gamma(i \Rightarrow f+\gamma) = \frac{4\alpha_{em} e_Q^2}{3}(2J' + \ell) S_{i,f} k^3 |I_{i,f}|^2 \qquad (7.1)$$

where

$$I_{i,f}(\lambda) = \int_0^\infty dr r^2 \Psi^*_{n_2\ell_2 m_2} r^\lambda \Psi_{n_1\ell_1 m_1} \qquad (7.2)$$

and we have

$$\begin{aligned} I_{i,f}(\lambda) &= \pi^{\rho(\ell_f-\ell_i)}\left[\frac{\Gamma(1+\rho+2\rho\ell_i)}{\Gamma(3\rho+2\rho\ell_i)}\cdot\frac{\Gamma^2(3\rho+\rho\ell_f+\rho\ell_i+\lambda\ell_f)}{\Gamma(1+\rho+\rho\ell_f)\Gamma(+3\rho+\rho\ell_f)}\right]^{1/2}\times \qquad (7.3) \\ &\times \frac{1}{\sigma^{\lambda/2}}\left(1+\frac{z_i-z_f}{2z_f}\right)^{-3\rho-\rho\ell_i-\rho\ell_f-\lambda\rho} z_i^{(3+2\ell_i)/2}\cdot z_f^{-(5+2\ell_f)/2}\,. \end{aligned}$$

**Table 7.** The width of radiative transition of meson.

| $1^3P_0 \to 1^3S_1$ | | $k\ Mev$ | $\Gamma\ keV$ |
|---|---|---|---|
| $(c\bar{c})$ | [2] | 305 | 105.2 |
| | our | 308.6 | 115.9 |
| $(b\bar{b})$ | [3] | 391 | 29.5 |
| | our | 330.5 | 33 |
| $(c\bar{b})$ | [4] | 370 | 43 |
| | our | 357.6 | 43.5 |

Let us consider the leptonic decay of vector mesons

$$\Gamma(V \Rightarrow e^+e^-) = \frac{4\pi\alpha_{em}^2}{m_q^2} Q_q^2 |\Psi_0(0)|^2 \ , \quad (7.4)$$

where

$$|\Psi_0(0)|^2 = \frac{1}{4\pi} \cdot \frac{z\sigma^{3/2}}{\rho\Gamma(3\rho)} \quad (7.5)$$

For the width of leptonic decay of vector mesons we have:

**Table 8.** The width of leptonic decay of vector mesons.

| $V \Rightarrow e^+e^-$ | $Exp\ [4]\ Mev$ | $our\ keV$ |
|---|---|---|
| $\Psi \Rightarrow e^+e^-$ | $5.54 \pm 0.96$ | 4.6 |
| $\Phi \Rightarrow e^+e^-$ | $52.86 \pm 0.19$ | 2.74 |
| $\rho \Rightarrow e^+e^-$ | $7.04 \pm 0.06$ | 7.29 |
| $\Upsilon \Rightarrow e^+e^-$ | $1.34 \pm 0.018$ | 1.16 |

## 8. Conclusion

**On the basis of the obtained results the following conclusions can be made:**

*Our approach is based on the investigation of the asymptotic behaviour of the polarization* loop function for scalar charged particles in an external gauge field and we determined the interaction Hamiltonian including the relativistic corrections.The potential is determined by the contributions of a every possible type of Feynman diagrams with exchange of gauge fields.The mass spectrum of the bound state is analytically derived. The mechanism for arising of the constituent mass of the relativistic bound state forming particles is explained. The mass and the constituent mass of mesons is calculated taking into account relativistic corrections with spin-orbit interactions.