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Nonperturbative calculation for the Adler function with Borel resummation

O.P. Solovtsova

Gomel State Technical University, Belarus

Plan of Talk

– Introduction

– Model & Physical applications

– NP contributions: residual condensates

– The Borel form of D-function

– Summary

To the memory of Igor Solovtsov

(May 2007)

Introduction

In comparing theoretical predictions with experimental data, it is important to connect measured quantities with 'simplest' theoretical objects. The hadronic correlator $\Pi(q^2)$ and the corresponding Adler function $D(Q^2)$, that appear in the process of e^+e^- annihilation into hadrons and the inclusive decay of the τ lepton, can play the role of these objects

$$\Pi_{\mu\nu}(q^2) = i \int d^4x\, e^{iqx} \, \langle 0 \left| T V_\mu(x) V_\nu(0)^+ \right| 0 \rangle$$

$$\propto (q_\mu q_\nu - g_{\mu\nu}\, q^2)\, \Pi(q^2) \qquad V^\mu_{ij} = \bar{\psi}_j \gamma^\mu \psi_i$$

$$D(Q^2) \equiv -Q^2 \, \frac{d\,\Pi(-Q^2)}{d\,Q^2} \qquad Q^2 = -q^2 > 0$$

[in Euclidian (spacelike) region]

$$R(s) = \mathrm{Im}\Pi(s)/\pi$$

- Ratio of hadronic to leptonic τ-decay widths in the vector channel

$$\boldsymbol{R}_\tau^V = R^{(0)} \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + \frac{2s}{M_\tau^2}\right) R(s)$$

- 'Light' Adler function (constructed from τ-decay data)

$$\boldsymbol{D}(Q^2) = Q^2 \int_0^{\infty} ds \, \frac{R(s)}{(s+Q^2)^2}$$

The Adler D function is an analytic

function in the complex Q^2-plane with a cut along the negative real axis.

- Smeared function

$$\boldsymbol{R}_{\Delta}(s) = \frac{\Delta}{\pi} \int_0^{\infty} ds' \, \frac{R(s')}{(s-s')^2 + \Delta^2};$$

- Hadronic contribution to the anomalous magnetic moment of the muon

$$\boldsymbol{a}_{\mu}^{\mathrm{had}} = \frac{1}{3} \left(\frac{\alpha}{\pi}\right)^2 \int_0^{\infty} \frac{ds}{s} \, K(s) \, R(s)$$

- Hadronic contribution to the fine structure constant

$$\boldsymbol{\Delta}\alpha_{\mathrm{had}}^{(5)}(s) = -\frac{\alpha(0)}{3\pi} \, s \, \mathcal{P} \int_0^{\infty} \frac{ds'}{s'} \, \frac{R(s')}{s'-s}.$$

A common feature of all these quantities is that they are defined through the function $R(s)$ integrated with some other functions. By definition, all these quantities include an infrared region as a part of the interval of integration and therefore, they cannot be directly calculated within perturbative QCD.

At the low energy scale the standard PT cannot adequately describe quark-gluon systems and, to build up an adequate approach, a method must be supplemented by non-perturbative formalism. To this end, one usually applies the operator product expansion (OPE), in which new parameters, vacuum condensates, are appeared. These additional parameters are usually extracted from a phenomenological analysis of the QCD sum rules. The Borel transform is the useful mathematical method, which is used for QCD sum rule treatment.

METHOD

Nonperturbative method of performing QCD calculations based on

1) the idea of variational perturbation theory (VPT)

in QCD the a-expansion method — I. Solovtsov

PLB 327 and 340 (1994)

2) taking into account the nonperturbative character of the light quark masses

3) involving a summation of infinite number of threshold singularities

[Milton and Solovtsov (2001)]

was applied for a description of above physical quantities.

The method of VPT leads the following connection between the expansion parameter a and the original coupling constant

$$\lambda = \frac{g^2}{(4\pi)^2} = \frac{\alpha_s}{4\pi} = \frac{1}{C}\,\frac{a^2}{(1-a)^3}\,.$$

For all values of the running coupling $\lambda \geq 0$ the expansion parameter a obeys the inequality $0 \leq a < 1$. The positive parameter C plays the role of a variational parameter.

An important feature of VPT approach is the fact that for sufficiently small value of the running coupling the a-expansion reproduces the standard perturbative expansion, and, therefore, the perturbative high-energy physics is preserved. In moving to low energies, where the standard PT breaks down, the parameter a remains small and we still stay within the region of applicability of the a-expansion method.

The Q^2-evolution of the expansion parameter $a(Q^2)$ is defined by renormalization group equation

$$\ln \frac{Q^2}{Q_0^2} = \frac{C}{2\,\beta_0}\,[\,f(a) - f(a_0)\,]\,,$$

where $a_0 = a(Q_0^2)\,,\ \beta_0 = 11 - 2/3\,N_f$ is the first coefficient of the standard β-function in perturbative expansion, N_f is the number of active quarks, and

$$f(a) = \frac{2}{a^2} - \frac{6}{a} - 48\,\ln\,a - \frac{18}{11}\,\frac{1}{1-a} + \frac{624}{121}\,\ln\,(1-a) + \frac{5184}{121}\,\ln\,(1+\frac{9}{2}\,a)$$

with an accuracy $O(a^3)$.

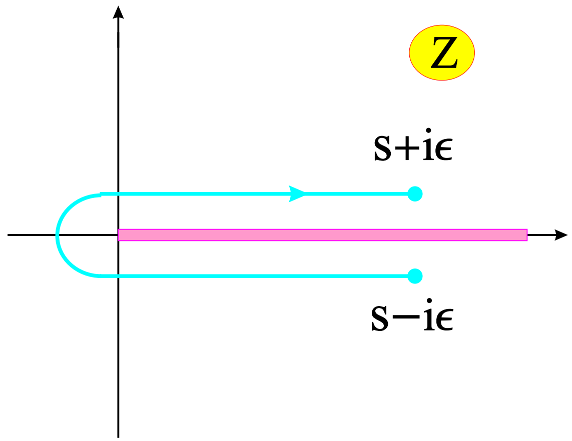
The function $f(a)$ is a *monotonous* in the interval $(0,1)$ and, therefore, for any values of Q^2, the equation has a unique solution in the interval $0 < a(Q^2) < 1\,$.

The parameter C can be defined, if one takes into account the Källén–Lehmann analyticity of the running parameter $a(Q^2)$.

$$R(s) = \left[1 + r_0 \lambda_s^{\text{eff}}(s)\right], \quad D(Q^2) = \left[1 + d_0 \lambda^{\text{eff}}(Q^2)\right] \quad (r_0 = d_0 = 4)$$

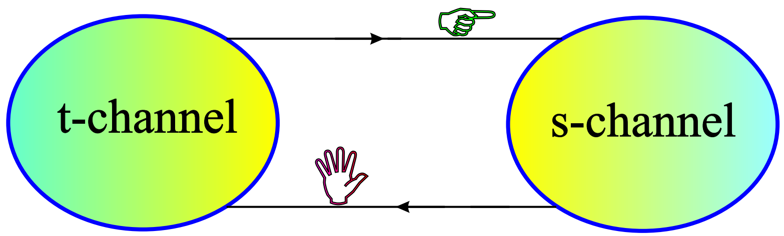
$$\lambda^{\text{eff}}(q^2) = -q^2 \int_0^\infty \frac{d\,s}{(s-q^2)^2}\, \lambda_s^{\text{eff}}(s),$$

$$\lambda_s^{\text{eff}}(s) = -\frac{1}{2\pi \mathrm{i}} \int_{s-\mathrm{i}\,\epsilon}^{s+i\epsilon} \frac{d\,z}{z}\, \lambda_t^{\text{eff}}(z).$$



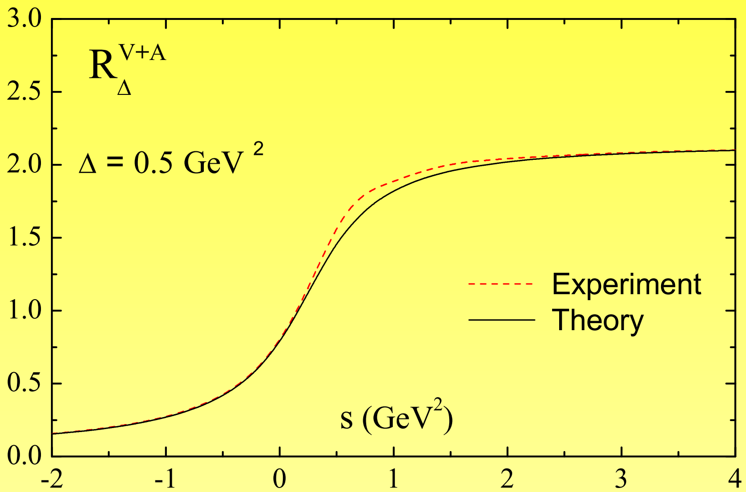
To parameterize $R(s)$ in terms of QCD parameters, a procedure of analytic continuation from the Euclidean to Minkowskian region is required.

The VPT allows one to perform this analytic continuation self-consistently.

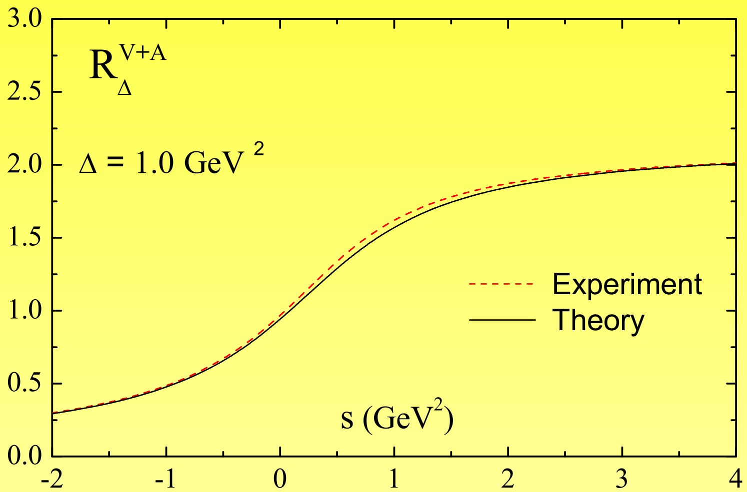


From the published in the Proc. of the XVIII Baldin Seminar – 2006 the paper of I. Solovtsov one can see that the proposed method allows us to describe the R-related quantities rather well.

For example, the value of the hadronic contribution to the anomalous magnetic moment is $a_\mu^{\rm had} = (702 \pm 16) \times 10^{-10}$.

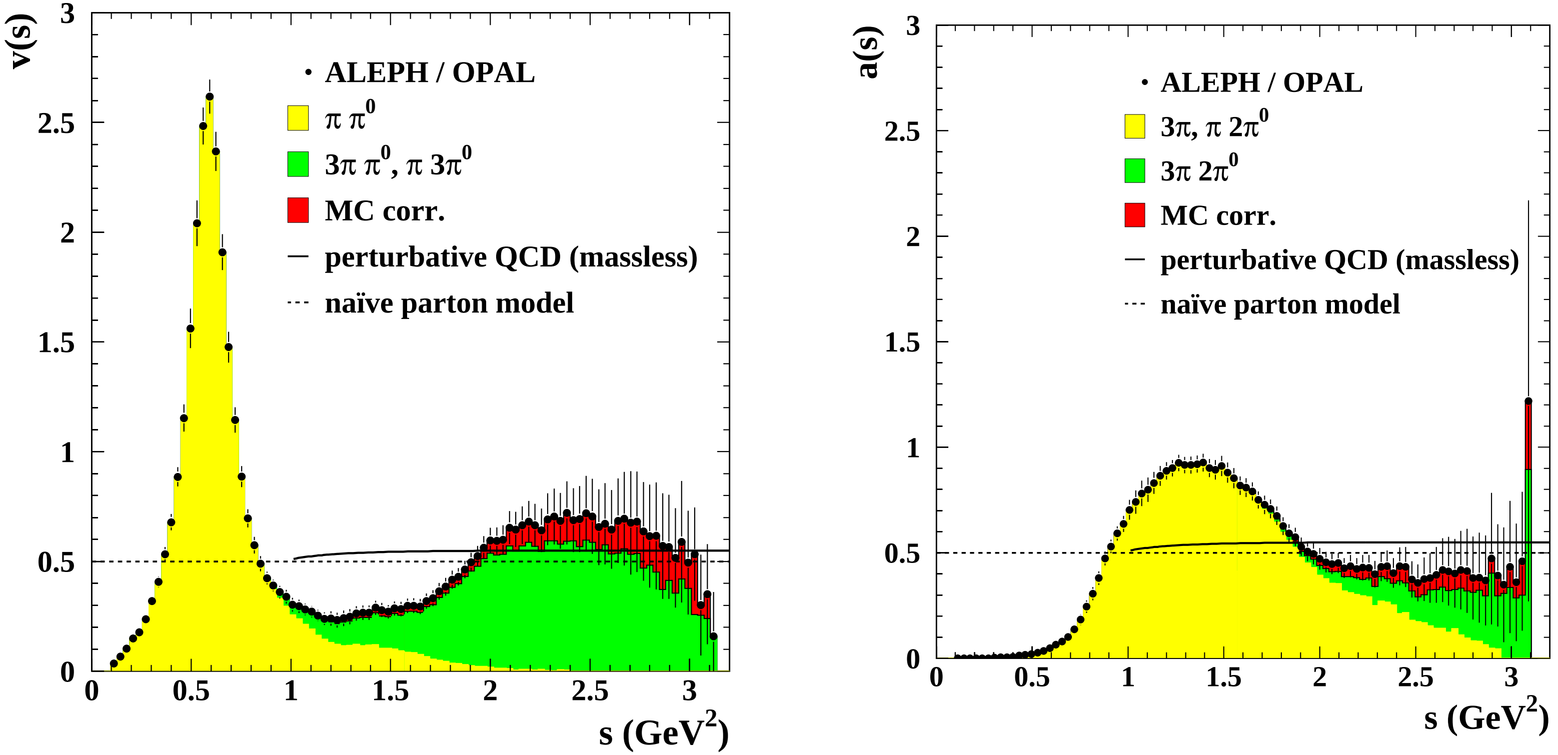


Vector + Axial-vector smeared function $R_\Delta^{V+A}(s)$ for $\Delta = 0.5$ GeV^2.



Vector + Axial-vector smeared function $R_\Delta^{V+A}(s)$ for $\Delta = 1.0$ GeV^2.

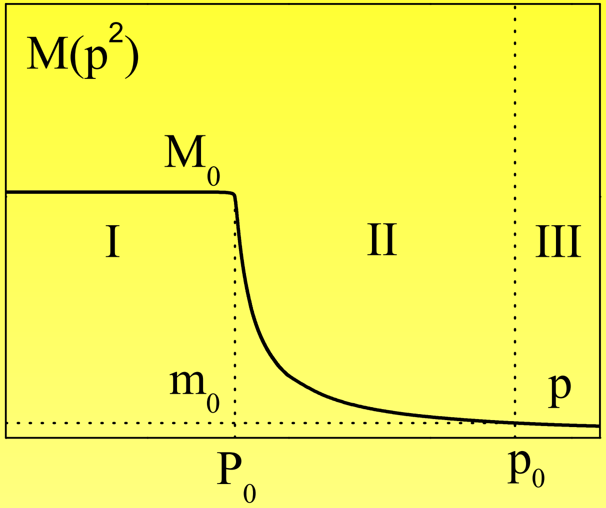
Experimental spectral functions: V and A channels



$$v(s)/a(s) = \frac{1}{2} R_{V/A}$$

Further investigations

The model of light quarks mass function



According to this model at small p^2 the function is rather smooth ($M_0 \simeq 260$ MeV). In the region $p^2 > 1 \div 2$ GeV^2 the principle behavior is defined by PT. The parameter m_0 is taken from the known values of the running (current) masses at $p_0 = 2$ GeV.

$f_\pi \ (\pi \to \mu\nu)$ $\qquad f_\pi^{\,\mathrm{exp}} = 92.40 \pm 0.26 \ \mathrm{MeV}$

Pagels-Stokar expression

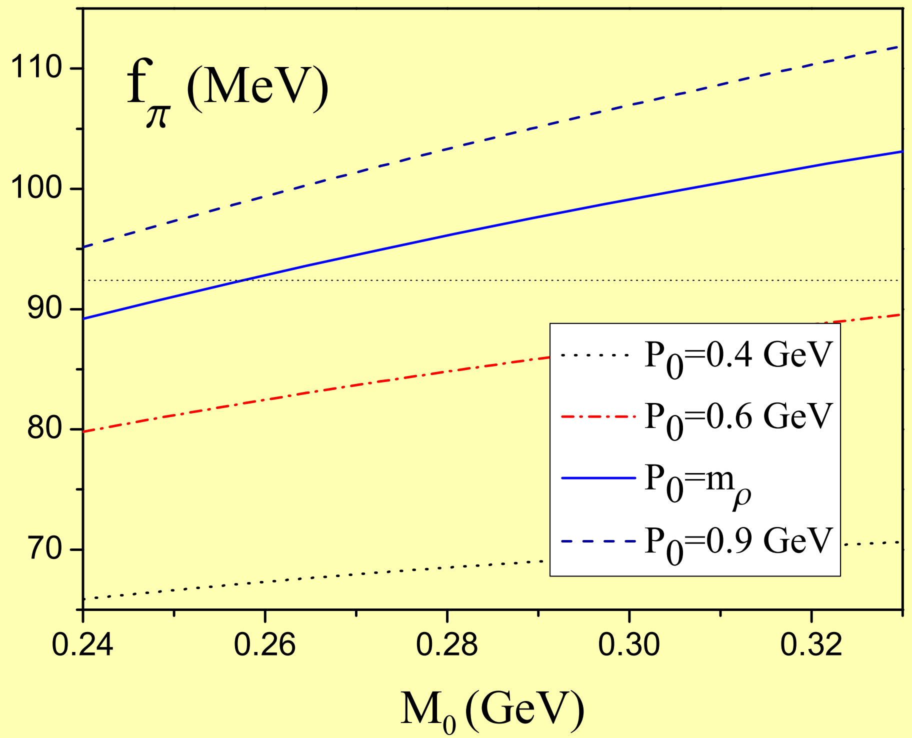
$$f_\pi^2 = \frac{3}{4\pi^2} \int\limits_0^\infty dp^2 \, \frac{p^2 \, M(p^2)}{[p^2 + M^2(p^2)]^2} \left[M(p^2) - \frac{p^2}{2} \frac{dM(p^2)}{dp^2} \right]$$

$$f_\pi^2 = f_{\pi,I}^2 + f_{\pi,II}^2 + f_{\pi,III}^2$$

$f_{\pi,I}^2$ – dominant

$f_{\pi,II}^2$ – near by 20%

$f_{\pi,III}^2$ – numerically small



So, additional information about f_π allows to get the restrictions on parameters of mass function (P_0 close to value of mass ρ-meson).

The quark condensate $\langle \bar{q}q \rangle$

$$\langle \bar{q}q \rangle_a = -\frac{N_c}{4\pi^2} \int_0^{a^2} dp^2 \, \frac{p^2 \, M(p^2)}{p^2 + M^2(p^2)}$$

At the same parameters of mass function ($M_0 = 0.26$ GeV, $P_0 = m_\rho$, $m_0 = 4$ MeV and $p_0 = 2$ GeV), we get

$$\langle 0|\bar{q}q|0\rangle|_{p_0^2} = -(235 \text{ MeV})^3 \Rightarrow \langle 0|\bar{q}q|0\rangle|_{\mu^2=1} = -(220 \text{ MeV})^3$$

rather well agreement with phenomenological value

$$\langle 0|\bar{q}q|0\rangle = -(240 \text{ MeV})^3$$

and coming from the Gell-Mann–Oakes-Renner relation

$$\langle 0|\bar{q}q|0\rangle = -\frac{m_\pi^2 \, f_\pi^2}{2\, m_0} \Rightarrow \langle 0|\bar{q}q|0\rangle = -(210 \text{ MeV})^3$$

Connection with Operator Product Expansion

The vacuum condensates play an important role in the QCD sum rule method [Shifman, Vainshtein, Zakharov (1979)]

Formally, the operator product expansion of the correlator $\Pi(q^2)$ of the vector quark current can be written as a sum of perturbative and nonperturbative parts

$$\Pi_{\mathrm{OPE}}(-Q^2) = \Pi_{\mathrm{PT}}(-Q^2) + \frac{\langle \mathcal{O}_2 \rangle}{Q^2} + \frac{\langle \mathcal{O}_4 \rangle}{Q^4} + \frac{\langle \mathcal{O}_6 \rangle}{Q^6} + \cdots$$

$\mathcal{O}_{2n}$ are the local operators constructed from quark and gluon fields.

Within the standard QCD sum rules approach $\langle \mathcal{O}_2 \rangle = 0$ since there is no gauge invariant operator of the dimension two.

The values of condensates, which are usually employed in phenomenological applications, are

$$\langle \mathcal{O}_4 \rangle_{\mathrm{phen}} \simeq 0.04\,\mathrm{GeV}^4, \quad \langle \mathcal{O}_6 \rangle_{\mathrm{phen}} \simeq -0.06\,\mathrm{GeV}^6.$$

Residual condensates

The difference between the experimental values and theoretical prediction of the correlator $\Pi(q^2)$ can be represented in the following form

$$\Delta\Pi(Q^2) = \Pi_{\rm expt}(Q^2) - \Pi_{\rm theor}(Q^2) \quad \Leftarrow \textit{ measure of knowledge/ignorance}$$

Dispersion relation

$$\Delta\Pi(Q^2) = \int_0^\infty ds\, \frac{\Delta R(s)}{s+Q^2}, \qquad \Delta R(s) = R_{\rm expt}(s) - R_{\rm theor}(s)$$

If one is able to derive the function $R_{\rm theor}$ exactly, in this case $\Delta R(s) = 0$. For approximate form of the quark current correlator $\Delta\Pi(-Q^2) \neq 0$. Nevertheless, one may expect that the model, which adequately describes the strong interaction processes in the infrared domain, inherently incorporates the most important condensates.

Assuming that the perturbative part of the correlator is reproduced good enough, one can represent the function $\Delta\Pi(-Q^2)$ as the sum of the residual nonperturbative contributions

$$\Delta\Pi(-Q^2) = \Pi_{\mathrm{expt}}(-Q^2) - \Pi_{\mathrm{theor}}(-Q^2) = \frac{O_2}{Q^2} + \frac{O_4}{Q^4} + \frac{O_6}{Q^6} + \cdots$$

Obviously, in this expression the "residual condensates" O_{2k} differ from above $\langle \mathcal{O}_{2k} \rangle$.

Similarly to the sum rule method, we apply the Borel transform, that eventually results in the basic sum rule

$$O_2 + \frac{O_4}{M^2} + \frac{O_6}{2M^4} = \mu_1(M^2),$$

where the Borel moments are

$$\mu_n(M^2) = \int_0^{s_0} ds\, s^{n-1} \exp(-s/M^2)\, \Delta R(s).$$

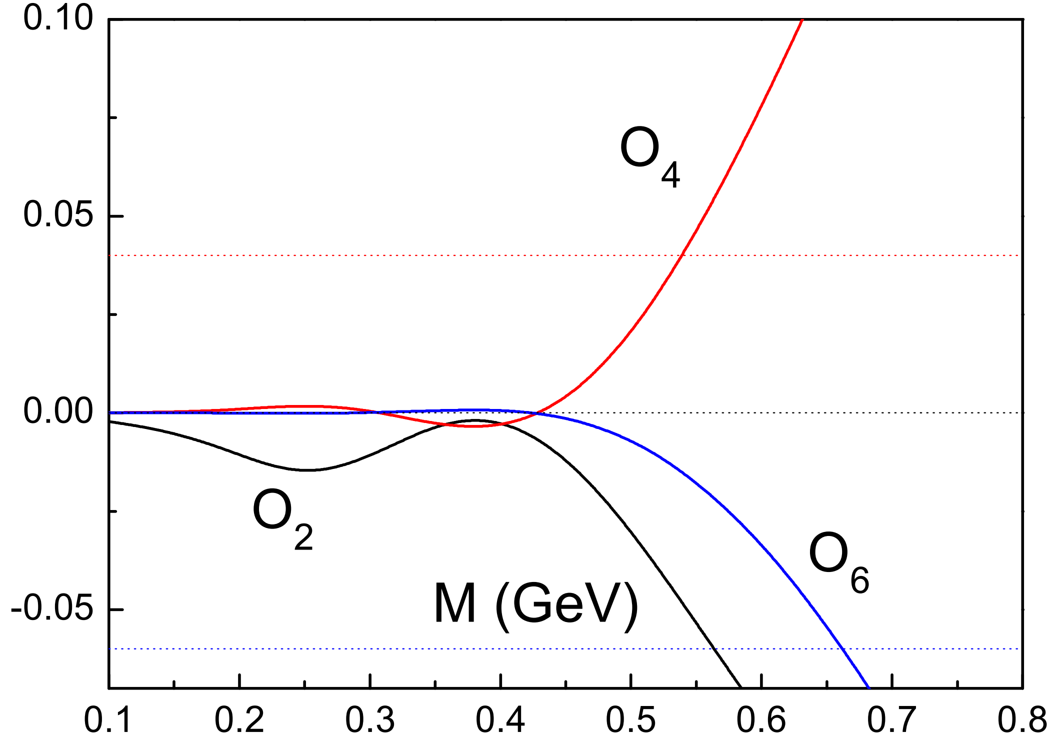
System of sum rules (can be derived by differentiating) gives the solution for the lowest dimension condensates O_2, O_4, and O_6:

$$O_2 = -2\,\mu_1(M^2) - \frac{\mu_2(M^2)}{2M^2} + 3\,\tilde{\mu}(M^2),$$
$$O_4 = 6\,M^2\mu_1(M^2) + 2\,\mu_2(M^2) - 6\,M^2\tilde{\mu}(M^2),$$
$$O_6 = -6\,M^4\mu_1(M^2) - 3\,M^2\mu_2(M^2) + 6\,M^4\tilde{\mu}(M^2),$$

where

$$\tilde{\mu}(M^2) = \int\limits_0^{s_0} ds\,\frac{1-\exp(-s/M^2)}{s/M^2}\,\Delta R(s).$$

Similarly to the sum rule method, the Borel parameter M plays the role of the optimization parameter.

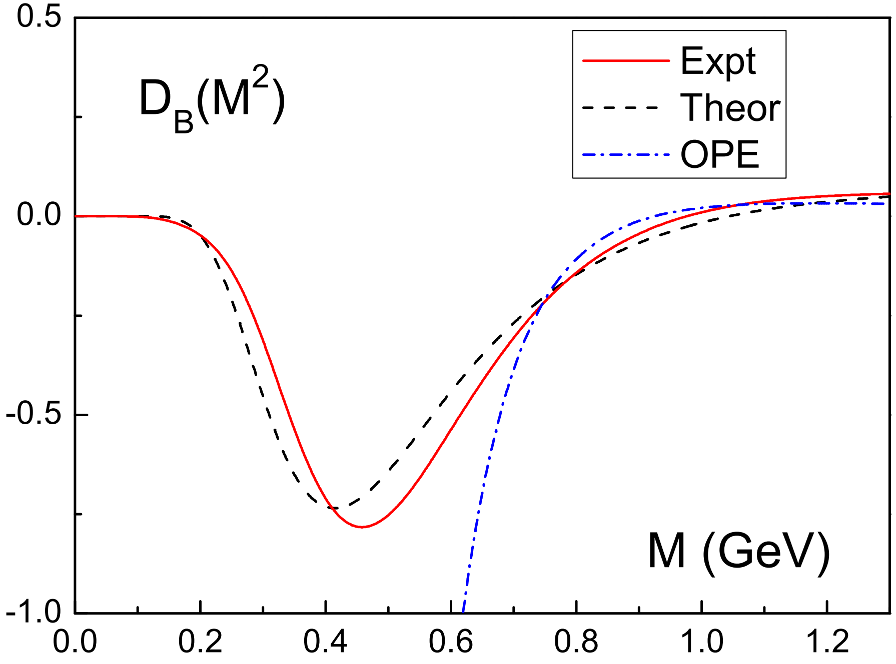


Horizontal lines correspond to the phenomenological values

$$\langle \mathcal{O}_4 \rangle_{\rm phen} \simeq 0.04\,{\rm GeV}^4, \quad \langle \mathcal{O}_6 \rangle_{\rm phen} \simeq -0.06\,{\rm GeV}^6.$$

Optimal values of O_{2n} are compatible with zero. The model developed essentially accumulates the basic condensates of the lowest dimension.

The result for the Borel form of the D-function



An "essential dynamics" of the function $D_{\rm B}(M^2)$ is observed in the region of ρ-meson mass and less. Our model describes experimental curve rather well for a whole interval of the Borel parameter M.

Summary

It was continued investigations in the framework of the nonperturbative method in QCD which on essences has been proposed by I. Solovtsov. This approach works well and gives stable results down to low energy scale.

It was shown that a behavior of the model of light quarks mass function leads to a good agrement with experimental value of weak decay constant of the pion, f_π, and with phenomenological value of the chiral condensate.

The Borel type sum rules, which allow us to determine the residual condensates, have been constructed. It was shown that within the method suggested the optimal values of these lower dimension condensates are close to zero. Therefore, the model includes at least important from phenomenological point of view condensates of lowest dimensions.

Any finite order of the operator product expansion fails to describe the infrared tail of the $D_{\rm B}$-function. Our result for the Borel form of the D-function gives a good agreement with experimental data down to lowest scales of the Borel parameter M^2. The approach, which we used here, is somewhat alternative to the OPE approach. Our method does not involve into consideration additional parameters and operates with parameters of the Lagrangian.

The End