# Low Energy Constants of $\chi$PT from the instanton vacuum

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# Outline

The QCD lagrangian has a form

$$\mathcal{L}_{\rm QCD} = -\frac{1}{2}\mathrm{tr}(G_{\mu\nu}G^{\mu\nu}) + \sum_{f=1}^{N_F}\bar{q}_f\left(i\widehat{D} - m_f\right)q_f$$

- Is invariant *w.r.t.* color gauge transformations $q \longrightarrow e^{i\vec{\alpha}\vec{t}_c}q$
- Is invariant *w.r.t.* flavour gauge transformations $q \longrightarrow e^{i\vec{\alpha}\vec{t}_f}q$
- In chiral limit there is an additional invariance *w.r.t.* nonsinglet axial flavour gauge transformations $q \longrightarrow e^{i\vec{\beta}\vec{t}_f\gamma_5}q$.
- The chiral symmetry is dynamically broken and leads to appearance of the Goldstones (mesons).
- Also, S$\chi$SB leads to nonzero vacuum condensates, such as $\langle\bar{q}q\rangle, \langle G^{\mu\nu}G_{\mu\nu}\rangle$, as well as masses of the baryons etc.
- For the lightest mesons with masses $M_f^2 \ll 1\,GeV^2$ the chiral symmetry must be a good approximation.

The basic object we study are the correlators. At low-energies the dynamics is described in terms of the effective degrees of freedom (mesons).

- Effective chiral lagrangian – description in terms of effective (meson) degrees of freedom. To the lowest order in pion momenta $q$ and external field $\hat{V} = s + p\gamma_5 + v_\mu\gamma_\mu + a_\mu\gamma_\mu\gamma_5$ it has a form (Gasser, Leutwyler, 1984)

$$L_2 = \frac{F^2}{2}\left\langle D_\mu U^T D_\mu U \right\rangle + F^2 \left\langle \chi^T U \right\rangle .$$

  $(U = u_0 + i\vec{\tau}\vec{u},\ U^\dagger U = 1, D_\mu u_0 = \partial_\mu u_0 + a^i_\mu u_i, D_\mu u_i = \partial_\mu u_i - a^i_\mu u_0 + \epsilon_{ijk} v^j_\mu u_k, \chi = 2B(s,\vec{p})$, consider $N_f = 2$.)
- The simplest observables: $\langle qq(m)\rangle = \frac{\delta \ln Z}{\delta s} \approx -F^2 B + \mathcal{O}(m)$, $\int d^4x\, e^{-iq\cdot x} \left\langle j^{a,5}_\mu(x) j^{b,5}_\nu(0)\right\rangle = \int d^4x\, e^{-iq\cdot x} \frac{\delta^2 \ln Z}{\delta a^a_\mu \delta a^b_\nu} = F_\pi^2 \delta^{ab}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2+M_\pi^2}\right) + \mathcal{O}(q^2),\ M_\pi^2 \approx 2B\, m + \mathcal{O}(m)$.
- The constants $F, B$ define pion decay constant and quark condensate in the chiral limit.

- $$L_4 = l_1(D_\mu U^T D_\mu U)^2 + l_2(D_\mu U^T D_\nu U)(D_\mu U^T D_\nu U) + l_3(\chi^T U)^2 + l_4(D_\mu \chi^T D_\mu U) + l_5(U^T F_{\mu\nu} F_{\mu\nu} U) + l_6(D_\mu \chi^T F_{\mu\nu} D_\nu U) + l_7(\tilde{\chi}^T U)^2 + h_1(\chi^T \chi) + h_2 \mathrm{tr}(F_{\mu\nu} F_{\mu\nu}) + h_3(\tilde{\chi}^T \tilde{\chi})$$
- So now we have 10 independent constants.
- $l_i, h_i$ are *bare* constants, they are renormalized by pion loops to $l_i^r(\mu^2)$.
- Physical observables should be expressed in terms of $\bar{l}_i = \frac{32\pi^2}{\gamma_i} l_i^r(\mu^2 = M_\pi^2)$. The $\mu^2$-dependence in $l_i^r$ (and consequently $M_\pi^2$-dependence in $\bar{l}_i$) are logarithmic, $\bar{l}_i = \alpha_i - \ln M_\pi^2$.

- For physical observables it leads to nonanalytical *m*-dependence (Novikov *et.al.*, 1981):

$$\langle \bar{q}q(m)\rangle = \langle \bar{q}q(0)\rangle \left(1 - \frac{3m_\pi^2}{32\pi^2 F^2}\ln m_\pi^2\right)$$

$$F_\pi^2(m) = F_\pi^2(0)\left(1 - \frac{m_\pi^2}{8\pi^2 F^2}\ln m_\pi^2\right)$$

$$M_\pi^2(m) = m_\pi^2\left(1 + \frac{m_\pi^2}{32\pi^2 F^2}\ln m_\pi^2\right)$$

# LECs and observables in pion physics:

Universality of constants. Example of observables (Gasser, Leutwyler, 1984):

1. $\pi - \pi$ S-wave scattering length:

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2}\left[1 + \frac{5M_\pi^2}{84\pi F_\pi^2}(\bar{l}_1 + 2\bar{l}_2 - \frac{3}{8}\bar{l}_3 + \frac{21}{10}\bar{l}_4 + \frac{21}{8})\right]$$

2. Pion electromagnetic charge radius:

$$F_v(t) = 1 + \frac{1}{6}t\left\langle r_\pi^2\right\rangle_v + ...,\ \left\langle r_\pi^2\right\rangle_v = \frac{1}{16\pi F}(\bar{l}_6 - 1) + O(m_\pi^2)$$

3. $\pi \rightarrow e\nu\gamma$ decay amplitude has a part $\sim (\bar{l}_6 - \bar{l}_5)$.
4. Pion electromagnetic polarizabilities ($\gamma\pi \rightarrow \gamma\pi$ process) also are $\sim (\bar{l}_6 - \bar{l}_5)$.

# Running and discontinued experiments

1. DIRAC@CERN ⇒ lifetime of $\pi^+\pi^-, \pi K$ atoms. $\Rightarrow |a_0^0 - a_0^2|$ and $|a_0^{1/2} - a_0^{3/2}|$ in $S$-channel up to 5% (Gasser *et.al.*, 2001, J. Schweizer, 2004).
2. $K \longrightarrow \pi\pi e\nu$ @BNL E865.$\Rightarrow a_0^0$.
3. $K^\pm \longrightarrow \pi^\pm\pi^+\pi^-$ @NA48/2.$\Rightarrow |a_0^0 - a_0^2|$.
4. $\gamma p \longrightarrow \gamma\pi^+ n$ reaction study at the Mainz Microtron MAMI to find pion electromagnetic polarizabilities.
5. (Discontinued) $\gamma\gamma \longrightarrow \pi^+\pi^-$ experiments as PLUTO, DM1, DM2
6. Lattice evaluation of different constants (MILC, ETM, JLQCD, RBC/UKQCD, PACS-CS, *etc.*)

We assume that the promising method is the application of instanton vacuum model.

# QCD instantons

Instantons –classical solutions of the equations of motion in Euclidean space. In singular gauge (Belavin *et.al.*, 1975):

$$A_\mu^{I,a}(x) = \frac{2\rho^2 \bar{\eta}^{\nu}_{\mu a}(x-z)_\nu}{(x-z)^2[\rho^2+(x-z)^2]}\,.$$

For the antiinstanton just change the t'Hoft symbol $\bar{\eta} \to \eta$.

- The solutions are (anti)self-dual, *i.e.* $G^a_{\mu\nu} = \pm \tilde{G}^a_{\mu\nu}$.
- The topological charge $Q = \frac{1}{32\pi^2}\int d^4x\, G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} = +1$ for instantons and $-1$ for antiinstantons.
- The action on both instantons and antiinstantons $S_I = \frac{8\pi^2}{g^2} \Rightarrow$ the amplitude of tunneling $\sim \exp\left(-S_I\right)$ with $|\Delta N_W| = 1$,
  $N_W = \frac{1}{24\pi^2}\int d^3x \epsilon_{ijk}\left\langle \left(U^\dagger\partial_i U\right)\left(U^\dagger\partial_j U\right)\left(U^\dagger\partial_k U\right)\right\rangle$.
- Number of collective coordinates for each instanton:

$$4\,(centre)\;+\;1\,(size)\;+\;(4N_c-5)\,(orientations)\;=\;4N_c$$

# Dependence on $N_{CS}$

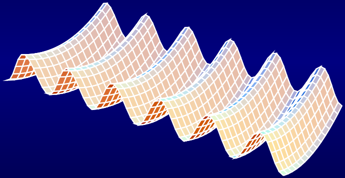


Figure: Dependence of the vacuum gluon fields energy on the Chern-Simons number $N_{CS}$.

# Instanton ensemble

- Sum ansatz $A = \sum_I A^I + \sum_{\bar{I}} A^{\bar{I}}$ for dilute gas approximation. Allows analytical evaluation, even with quarks.
- Example of exact multiinstanton solution (self-duality):

$$A^a_\mu = \bar{\eta}_{a\mu\nu}\partial_\nu \ln\left(1 + \sum_i \frac{\rho_i^2}{(x - z_i)^2}\right)$$

- Instanton-antiinstanton interactions: Ratio ansatz, Streamline ansatz. *Sum ansatz gives too strong repulsion for $R \leq \rho$.*
- Partition function–only numerically (lattice).

# Parameters of instanton ensemble

- Size distribution $D(\rho)$ and average value $\bar{\rho}$
- Density of instantons (or average interinstanton distance $\bar{R}$)
- Results:
  - Lattice estimate: $\bar{R} \approx 0.89\, fm$, $\bar{\rho} \approx 0.36\, fm$,
  - Phenomenological estimate: $\bar{R} \approx 1\, fm$, $\bar{\rho} \approx 0.33\, fm$,
  - Our estimate (with account of $1/N_c$ corrections): $\bar{R} \approx 0.76\, fm$, $\bar{\rho} \approx 0.32\, fm$,

  Thus within $10 - 15\%$ uncertainty different approaches give similar estimates
- Packing parameter $\underline{\pi^2 (\frac{\bar{\rho}}{\bar{R}})^4 \sim 0.1 - 0.3}$

  $\Rightarrow$ Independent averaging over instanton positions and orientations.

# QCD vacuum on the lattice

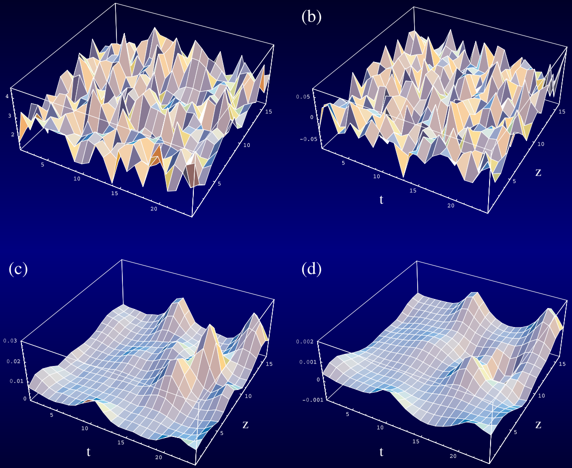


Figure: Action and topological charge densities in different configurations on the lattice.

# Light quarks in the instanton background

Basic assumptions (Diakonov *et.al.*, 1986-2006):

- Sum ansatz as background. Quarks ⇒ quenched approximation.
- Zero-mode approximation

$$S(x,y) \approx \frac{|\Phi_0\rangle\,\langle\Phi_0|}{im} + \frac{1}{i\hat{\partial}}$$
$$(i\hat{\partial} + g\hat{A})\Phi_0 = 0,$$

- The number of colors $N_c \to \infty$, LO over $N_c$ is kept.
- The width of the size distribution is suppressed as $1/N_c$

are working well at $m \Rightarrow 0$ but wrong beyond the chiral limit.

# Zero mode vs. Chiral Symmetry

Extension of zero-mode approximation beyond the chiral limit:

$$S_i = S_0 - S_0 \hat{p} \frac{|\Phi_{0i}\rangle\langle\Phi_{0i}|}{\langle\Phi_{0i}|\hat{p} S_0 \hat{p}|\Phi_{0i}\rangle} \hat{p} S_0, \;\; S_0 = \frac{1}{\hat{p} + im},$$

$$S_i|\Phi_{0i}\rangle = \frac{1}{im}|\Phi_{0i}\rangle, \;\; \langle\Phi_{0i}|S_i = \langle\Phi_{0i}|\frac{1}{im}.$$

Full propagator in the presence of the external fields $\hat{V} = s + p\gamma_5 + \hat{v} + \hat{a}\gamma_5$:

$$\tilde{S} - \tilde{S}_0 = -\tilde{S}_0 \sum_{i,j} \hat{p}|\phi_{0i}\rangle \left\langle \phi_{0i} \left| \left( \frac{1}{\hat{p}\tilde{S}_0\hat{p}} \right) \right| \phi_{0j} \right\rangle \langle\phi_{0j}|\hat{p}\tilde{S}_0$$

$$|\phi_0\rangle = \frac{1}{\hat{p}} L \hat{p} |\Phi_0\rangle, \;\; \tilde{S}_0 = \frac{1}{\hat{p} + \hat{V} + im}$$

$$L_i(x, z_i) = \mathrm{P} \exp\left( i \int_{z_i}^{x} dy_\mu (v_\mu(y) + a_\mu(y)\gamma_5) \right)$$

# Effective action

$$\ln \tilde{\mathrm{Det}}_{low} = \mathrm{Tr}\int dm\, \tilde{S}(m) = \ln\det\langle\phi_{0,i}|\hat{p}\tilde{S}_0^{fg}\hat{p}|\phi_{0,j}\rangle,$$

Averaging of $\tilde{\mathrm{Det}}_{low}$ over instantons by means of fermionization → constituent quarks → partition function $Z$. Exponentiation in $Z$ via Stirling-like formula → dynamical coupling $\lambda$

$$Z_N = \int d\lambda_+ d\lambda_- D\bar{\psi} D\psi e^{-S}$$

$$S = N_\pm \ln\frac{K}{\lambda_\pm} - N_\pm + \psi^\dagger(i\hat{\partial} + \hat{V} + im)\psi + \lambda_\pm Y_2^\pm$$

$$Y_2^\pm = \int d\rho D(\rho)\left(\alpha^2 \det_f J^\pm + \beta^2 \det_f J^\pm_{\mu\nu}\right)$$

$$\frac{\beta^2}{\alpha^2} := \frac{1}{8N_c}\frac{2N_c}{2N_c-1} = \frac{1}{8N_c-4} = \mathcal{O}\left(\frac{1}{N_c}\right)$$

$$J^\pm_{fg} = \psi_f^\dagger \bar{L}\frac{1\pm\gamma_5}{2}L\psi_g,\ J^\pm_{\mu\nu} = \psi_f^\dagger \bar{L}\sigma_{\mu\nu}\frac{1\pm\gamma_5}{2}L\psi_g.$$

# Effective action after bosonization

$$Z_N = \int d\lambda_+ d\lambda_- D\bar\psi D\psi D\Phi^\pm D\Phi^\pm_{\mu\nu} e^{-S}$$

$$S = -N_\pm \ln \lambda_\pm + 2\left(\Phi_i^2 + \frac{1}{2}\Phi_{i,\mu\nu}^2\right) + \psi^\dagger \left[ i\hat\partial + \hat V + im + \right.$$

$$\left. i\lambda^{0.5}\bar L F(p)\left(\alpha\Phi_i\Gamma_i + \frac{1}{2}\beta\Phi_{i,\mu\nu}\sigma_{\mu\nu}\Gamma_i\right) F(p) L^{-1}\right]\psi$$

$\Gamma_i = \{(1, i\vec\tau\gamma_5), (\gamma_5, i\vec\tau)\}$

Integrate out fermions:

$$S = -N_\pm \ln \lambda_\pm + 2\left(\Phi_i^2 + \frac{1}{2}\Phi_{i,\mu\nu}^2\right) - Tr \log \left[ \hat p + \hat V + im + \right.$$

$$\left. i\lambda^{0.5}\bar L F(p)\left(\alpha\Phi_i\Gamma_i + \frac{1}{2}\beta\Phi_{i,\mu\nu}\sigma_{\mu\nu}\Gamma_i\right) F(p) L^{-1}\right]$$

- Bosonization $\Rightarrow$ mesons. Chiral doublets: $(\sigma, \vec{\phi})$, $(\eta, \vec{\sigma})$ and $(\sigma_{\mu,\nu}, \vec{\phi}_{\mu\nu})$.
- Meson loops are $1/N_c$ corrections $\Rightarrow$ need to take into account all $1/N_c$ corrections. $\Rightarrow$ Double expansion ($1/N_c$, $m$).
- Regularization@$q \sim \rho^{-1}$ via nonlocality.
- Other sources of $1/N_c$-correction:
  - Finite width of size distribution.
  - Shift of the coupling $\lambda$.

# Momentum dependence of dynamical quark mass $M(q)$

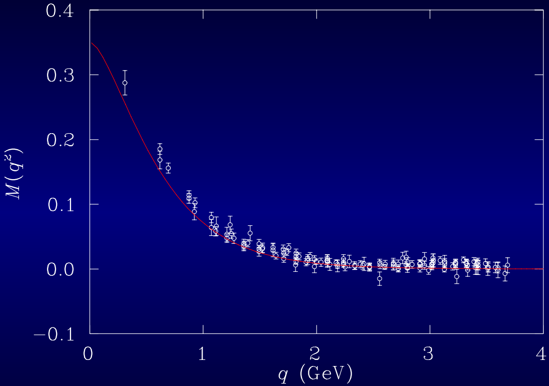


Figure: Momentum dependence of dynamical quark mass $M(q)$ in the chiral limit. Points: lattice result (P.Bowman *et. al.*,, 2004). Red line: zero-mode approximation (Diakonov&Petrov86), **no fitting**.

# Dynamical quark mass

$$\frac{N}{V} - \frac{1}{2V} Tr\left(Q(p)\right)$$
$$+ \int \frac{d^4q}{2\sigma^2(2\pi)^4} \sum_i \left(V_2^i(q) - V_3^i(q)\right) \Pi_i(q) = 0,$$

$$4\sigma^2 - \frac{1}{V} Tr\left(Q(p)\right) + \int \frac{d^4q}{\sigma^2(2\pi)^4} \sum_i V_3^i(q)\Pi_i(q) = 0.$$

$$Q(p) = \frac{iM(p)}{\hat{p} + i\mu(p)}, V_n^i(q) = Tr\left(Q^{n-1}(p)\Gamma_i Q(p+q)\Gamma_i\right)$$

Chiral log theorem:

$$M(m) = M(0)\left(1 - \frac{3m_\pi^2}{32\pi^2 F^2} \ln m_\pi^2\right)$$

# $m$-dependence of the dynamical quark mass $M(m)$.

$$M(m) = 0.36 - 2.36\, m - \frac{m}{N_c}(0.808 + 4.197 \ln m) + \mathcal{O}\left(m^2, \frac{1}{N_c}\right)$$

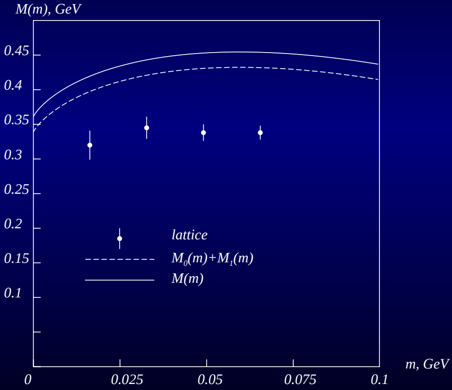


Figure: $m$-dependence of the dynamical quark mass $M(m)$. Comparison with lattice data (Bowman 2005)

# Finite width correction

2-loop instanton size distribution (Diakonov ' 83, Vainshtein *et.al.*, ' 82)

$$D(\rho) \sim (\Lambda\rho)^{\frac{11N_c}{3}-5}\left(\ln\left(\Lambda\rho\right)\right)^{-N_c\left(\frac{5}{11}-\frac{255}{1331\ln(\Lambda\rho)}\right)}$$

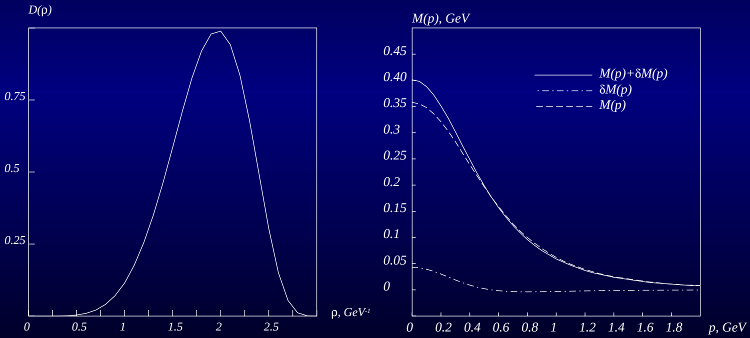
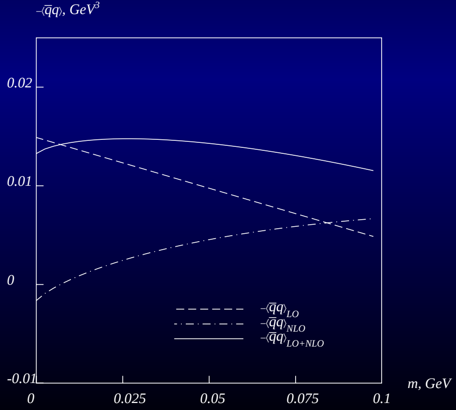


Figure: Left: Instanton size distribution. Right: change of the $M(p)$-dependence due to FWC.

# Quark condensate

$$-\langle\bar{q}q\rangle(m) = ((0.00497 - 0.0343\,m)\ N_c +$$
$$(0.00168 - 0.0494\,m - 0.0580\,m\ln m)) + \mathcal{O}\left(m^2, \frac{1}{N_c^2}\right)$$



# $F_\pi,\ M_\pi$

$\int d^4x\, e^{-iq\cdot x} \left\langle j_\mu^{a,5}(x) j_\nu^{b,5}(0) \right\rangle$-correlator

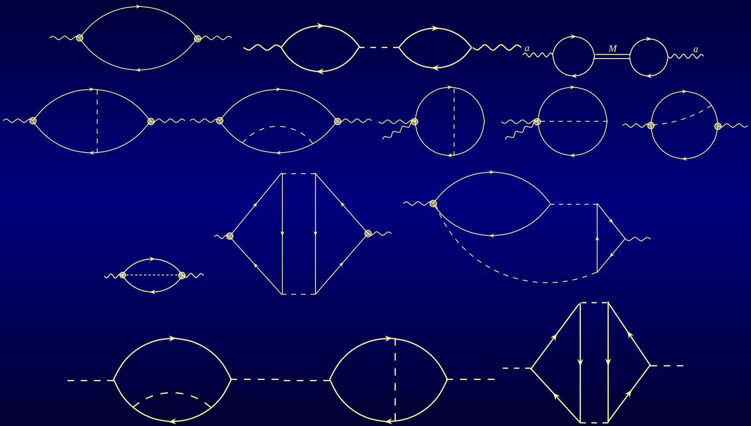


Figure: Contribution to correlator and $\pi$-meson propagator (last row)

# $F_\pi, \; M_\pi$

$$
\begin{aligned}
&F_\pi^2 = N_c \left( \left( 2.85 - \frac{0.869}{N_c} \right) - \left( 3.51 + \frac{0.815}{N_c} \right) m - \right. \\
&\left. - \frac{44.25}{N_c} \, m \ln m + \mathcal{O}(m^2) \right) \cdot 10^{-3} \; [GeV^2] = \\
&(7.67 - 11.35\, m - 44.25\, m \ln m) \cdot 10^{-3} \; [GeV^2] \\
&M_\pi^2 = m \left( \left( 3.49 + \frac{1.63}{N_c} \right) + \right. \\
&\left. m \left( 15.5 + \frac{18.25}{N_c} + \frac{13.5577}{N_c} \ln m \right) + \mathcal{O}(m^2) \right) = \\
&m\,(4.04 \; + 21.587\, m + 4.52\, m \ln m + \mathcal{O}(m^2))[GeV^2]
\end{aligned}
$$

# $F_\pi, \ M_\pi$

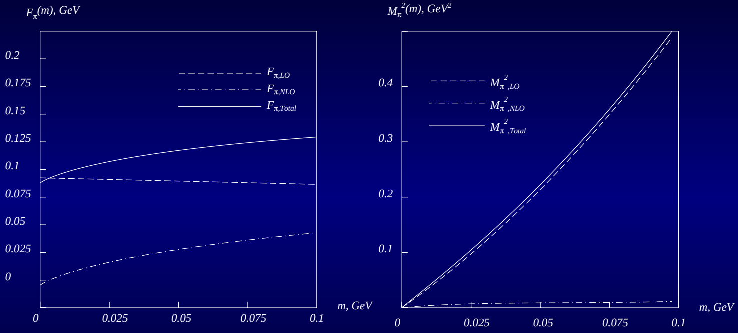


Figure: $F_\pi$ and $M_\pi^2$ as a function of $m$.

# $F_\pi, \; M_\pi$

- $F^2 = \left(2.85\, N_c - 0.87 + \mathcal{O}\left(\frac{1}{N_c}\right)\right) \times 10^{-3} [GeV^2]$

  $B = 1.75 + \frac{0.82}{N_c} + \mathcal{O}\left(\frac{1}{N_c^2}\right) [GeV]$

-

$$\bar{l}_3 = \frac{-1.14\, N_c \left(1 + \frac{0.872}{N_c} + \frac{0.875 \ln m}{N_c} + \mathcal{O}\left(\frac{1}{N_c^2}\right)\right)}{1 + \frac{0.94}{N_c} + \mathcal{O}\left(\frac{1}{N_c^2}\right)} =$$

$$-1.14\, N_c + 0.074 - \ln m + \mathcal{O}\left(\frac{1}{N_c}\right)$$

$$\bar{l}_4 = \frac{-0.079\, N_c \left(1 + \frac{0.232}{N_c} + \frac{12.6 \ln m}{N_c}\right)}{1 + \frac{0.47}{N_c} + \mathcal{O}\left(\frac{1}{N_c^2}\right)} =$$

$$-0.079\, N_c + 0.0187 - \ln m + \mathcal{O}\left(\frac{1}{N_c}\right)$$

# The low-energy constant $\bar{l}_3$.

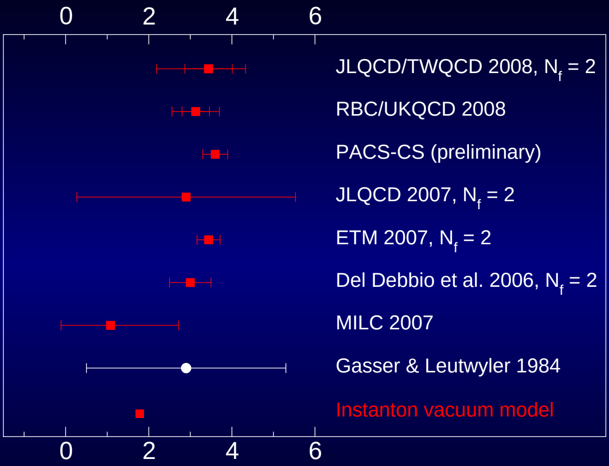


Figure: The low-energy constant $\bar{l}_3$: recent lattice results from different collaborations, phenomenological estimates from (Leutwyler 2008) and our result.

# The low-energy constant $\bar{l}_4$.

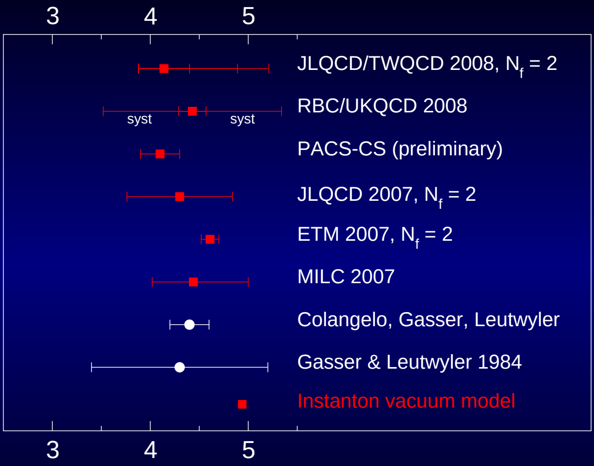


Figure: The low-energy constant $\bar{l}_4$: recent lattice results from different collaborations, phenomenological estimates from (Leutwyler 2008) and our result.

# $F_\pi, \ M_\pi$

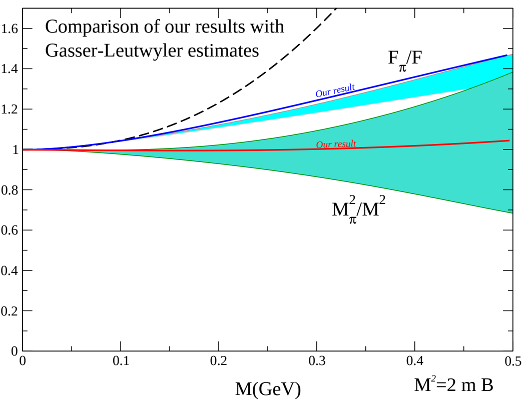


Figure: $m-$dependencies of $F_\pi$, $M_\pi$: comparison with phenomenological data from (Leutwyler 2001)

# Conclusion and outlook

- We established a reliable theoretical framework for evaluation the ChPT low-energy constants *with account of all $1/N_c$ corrections*. $1/N_c$ corrections are important, esp. for $\bar{l}_i$.
- We evaluated the $m$-dependence of $F_\pi$, $M_\pi$ and extracted the constants $\bar{l}_3$, $\bar{l}_4$. The found values are in reasonable agreement with lattice results and phenomenological estimates.
- The calculations of all other constants and the extension to the $N_f = 3$ case are on the way.