Renormdynamics and Scaling functions

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For Quantum field theory models, Renormdynamic equations of motion and their solutions are given. Universal scaling functions of High energy physics are considered. Explicit form of KNO function constructed.

Introduction

In the Universe matter has manly two geometric structures, homogeneous [1] and hierarchical one [2] . Homogeneous structures are naturally described by real numbers with usual archimedean metrics, hierarchical one are described with non-archimedean metrics (see e.g. [3]).

Discrete, finite, regularized, version of the homogenous structures are homogeneous lattices withe constant step, and distance rising as arithmetic progression. Discrete version of the hierarchical structures is hierarchical lattice-tree with scale rising in geometric progression.

There is an opinion that present-day theoretical physics needs (almost) all mathematics, and the progress of modern mathematics is stimulated by fundamental problems of theoretical physics.

Quantum field theory and Fractal calculus - Universal language of fundamental physics

Quantum field theory

As a concrete model, we take relativistic scalar field model with lagrangian (see e.g. [5])

$$L = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - g\varphi^n,\ \mu = 0, 1, ..., D-1 \tag{1}$$

In the case

$$n = \frac{2D}{D-2} \tag{2}$$

the coupling constant g is dimensionless, and the model is renormalizable. We take euklidian form of the QFT which unifies quantum and statistical physics problems. The main objects of theory are Green functions - correlation functions - correlators,

$$\begin{aligned} &G_m(x_1, x_2, ..., x_m) =< \varphi(x_1)\varphi(x_2)...\varphi(x_m) > \\ &= Z_0^{-1}\int d\varphi(x)\varphi(x_1)\varphi(x_2)...\varphi(x_m)e^{-S(\varphi)} \end{aligned} \tag{3}$$

where $d\varphi-$ invariant measure

$$d(\varphi + a) = d\varphi. \tag{4}$$

For gaussian actions,

$$S = S_2 = \int dxdy\phi(x)A(x,y)\phi(y) \tag{5}$$

the QFT is solvable,

$$\begin{aligned} G_m(x_1, ..., x_m) &= \frac{\delta^m}{\delta J(x_1)...J(x_m)} lnZ_J|_{J=0}, \\ Z_J &= \int d\varphi e^{-S_2+J\cdot\varphi} = e^{\frac{1}{4}\int dxdyJ(x)A^{-1}(x,y)J(y)} \end{aligned} \tag{6}$$

Non trivial problem is to calculate correlators for non gaussian QFT

Renormdynamics

In quantum perturbation calculations [6], we find the following corrections to the classical lagrangian

$$\Delta L = (z-1)\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - (z_m-1)\frac{m^2}{2}\varphi^2 - (z_g-1)g\varphi^n. \tag{7}$$

Corrected, effective, lagrangian become

$$L + \Delta L = z\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - z_m\frac{m^2}{2}\varphi^2 - z_g g\varphi^n \tag{8}$$

We can restor the (classical) form of the lagrangian, by corresponding renormalization (compensating) transformations,

$$\begin{aligned} &\varphi \Rightarrow z^{-1/2}\varphi \\ &m^2 \Rightarrow z_m^{-1} z m^2 \\ &g \Rightarrow z_g^{-1} z^{n/2} g \end{aligned} \tag{9}$$

So, if we order quantum correction in some discrete (or continual) way, we can include them step by step, which will be equivalent to the corresponding evolution equations for constants and fields [7]. These equations define the evolution from classical theory to quantum one. Frequently quantum corrections are ill defined, singular or divergent, so we need some regularization. For some field theory models, e.g. Youkawa nuklon-mezon model, quantum corrections invent new structures, in the case, mezon selfinterection, so quantum theory has extended structure. In this way, we can generate from classical Fermi like models the standard model of particle physics. If the structure elements of a (quantum)field theory model is finite, we have a renormalizable model, we may have also infinite number of structure elements, in the case of a quantum gravity model e.g.

In the infinitezimal form we have the following renormdynamic motion equations

$$\begin{aligned} &\frac{\mu d}{d\mu} g = \frac{d}{dt} g \equiv \dot{g} = \beta(g),\ t = \ln(\frac{\mu}{\mu_0}), \\ &\dot{m} = \eta(g) m, \\ &\dot{\varphi} = (\frac{\mu \partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \eta(g) \frac{m \partial}{\partial m}) \varphi \equiv D \varphi = -\frac{1}{2} \gamma(g) \varphi \end{aligned} \tag{10}$$

For correlators, renormdinamic equations are

$$(D + n \frac{\gamma(g)}{2}) G_n = 0, \tag{11}$$

For renorminvariant quantities - renomintegrals of motion I,

$$\dot{I} = DI = 0, \tag{12}$$

Solution of the renormdynamic equation for coupling constant, $\bar{g}$, is given in the implicit form by the following integral

$$\int_g^{\bar{g}} \frac{dg}{\beta(g)} = \ln \frac{\bar{\mu}}{\mu} \equiv t \tag{13}$$

The mass parameter running is given as

$$m = \bar{m} \exp\left(-\int_\mu^{\bar{\mu}} \frac{d\mu}{\mu} \eta(g(\mu))\right), \tag{14}$$

the correlator (renorm)dynamics is given as

$$G_n(p; g, m, \mu) = \exp\left(\frac{n}{2} \int_\mu^{\bar{\mu}} \frac{d\mu}{\mu} \gamma(g(\mu))\right) \cdot G_n(p; \bar{g}, \bar{m}, \bar{\mu}) \tag{15}$$

Nambu - Poisson formulation of Renormdynamics

In the case of several integrals of motion, $H_n,\ 1 \leq n \leq N$, we can formulate Renormdynamics as Nambu - Poisson dynamics (see e.g. [8])

$$\dot{\varphi}(x) = [\varphi(x), H_1, H_2, ..., H_N], \tag{16}$$

where φ is an observable as a function of the coupling constants $x_m,\ 1 \leq m \leq M$.

In the case of Standard model [9], we have three coupling constants, $M = 3$.

Renormdynamics of observable quantities in high energy physics

Let us consider one particle semiinclusive distribution

$$F(p,n) = \frac{d\sigma_n}{\bar{d}p} = \frac{1}{(n-1)!} \int \prod_{i=1}^{n-1} \bar{d}p_i' \delta(p_1 + p_2 - p - \Sigma_{i=1}^{n-1} p_i')$$
$$\cdot |G_{n+2}(p_1, p_2, p, p_1', p_2', ..., p_{n-1}'; g(\mu), m(\mu)), \mu)|^2,$$
$$\bar{d}p \equiv \frac{d^3p}{E(p)},\ E(p) = \sqrt{p^2 + m^2}. \tag{17}$$

From renormdynamic equation

$$DG_{n+2} = \frac{\gamma}{2}(n+2)G_{n+2}, \tag{18}$$

We obtain

$$\begin{aligned}
&DF(p,n) = \gamma(n+2)F(p,n),\\
&DF(p) = \gamma(<n>+2)F(p),\\
&D<n^k(p)>= \gamma(<n^{k+1}(p)> - <n^k(p)><n(p)>),\\
&DC_k = \gamma<n(p)>(C_{k+1} - C_k(1+k(C_2-1)))\\
&F(p) \equiv \frac{d\sigma}{\bar{d}p} = \sum_n \frac{d\sigma_n}{\bar{d}p},\ <n^k(p)>= \frac{\sum_n n^k d\sigma_n/\bar{d}p}{\sum_n d\sigma_n/\bar{d}p}\\
&C_k = \frac{<n^k(p)>}{<n(p)>^k}
\end{aligned} \tag{19}$$

Universal scaling relations for multi particle cross sections

From dimensional considerations, following combination of cross sections must be universal function (Koba, Nielsen, Olesen, 1972)[10]

$$< n > \frac{\sigma_n}{\sigma} = \Psi(\frac{n}{< n >}), \tag{20}$$

similar relation for the inclusive cross sections is (Matveev, Sissakian, Slepchenko, 1975) [11].

$$< n(p) > \frac{d\sigma_n}{\bar{d}p} / \frac{d\sigma}{\bar{d}p} = \Psi(\frac{n}{< n(p) >}) \tag{21}$$

Let us find explicit form of the universal functions from renormdynamic equations. From the definition of the moments we have,

$$C_k = \int_0^\infty dx x^k \Psi(x), \tag{22}$$

so they are independent from different parameters,

$$\begin{aligned} &\mathcal{D}C_k = 0 \Rightarrow C_{k+1} = (1 + k(C_2 - 1))C_k \Rightarrow \\ &C_k = (1 + (k-1)(C_2 - 1))...(1 + 2(C_2 - 1))C_2. \end{aligned} \tag{23}$$

Now we can invert momentum transform and find (see N.M. 1980, [5] and appendix) universal functions (Ernst, Schmitt, 1976; Slepchenko, Sissakian, N.M. 1978)[12], [13].

$$\Psi(z) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dn z^{-n-1} C_n = \frac{c^c}{\Gamma(c)} z^{c-1} e^{-cz},$$
$$C_2 = 1 + \frac{1}{c} \tag{24}$$

The value of parameter c can be measured from the dispersion low,

$$D = \sqrt{< n^2 > - < n >^2} = \sqrt{C_2 - 1} < n >= \frac{1}{\sqrt{c}} < n > . \tag{25}$$

$1/<n>$ correction to the scaling function

We can calculate also $1/<n>$ correction to the scaling function

$$
\begin{aligned}
&<n>\frac{\sigma_n}{\sigma}=\Psi=\Psi_0(\frac{n}{<n>})+\frac{1}{<n>}\Psi_1(\frac{n}{<n>}),\\
&C_k=C_k^0+\frac{1}{<n>}C_k^1,\\
&C_k^0=\int_0^\infty dxx^k\Psi_0(x),\ C_k^1=\int_0^\infty dxx^k\Psi_1(x),\\
&\Psi_1(z)=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}dnz^{-n-1}C_n^1=\frac{C_2^1c^2}{2}(z-2+\frac{c-1}{cz})\Psi_0
\end{aligned}
\tag{26}
$$

Closed equation of renormdynamics for the generating function of the observables

Let us consider generating function of the topological crossections

$$F(h,g,m,\mu)=\Sigma_{n\geq 2}h^n\sigma_n,$$
$$\sigma_n=\frac{1}{n!}\frac{d^n}{dh^n}F|_{h=0},$$
$$\sigma=F|_{h=1},\quad <n>=\frac{d}{dh}\ln F|_{h=1},... \tag{27}$$

It is natural that for generating function we have closed renormdynamic equation (N.M. 1980) [5]

$$(D-\gamma(\frac{h\partial}{\partial h}+2))F=0,$$
$$F(h,g,m,\mu)=F(\bar{h},\bar{g},\bar{m},\bar{\mu})\exp(2\int_{\mu}^{\bar{\mu}}\frac{d\mu}{\mu}\gamma(\bar{g}(\mu))),$$
$$\bar{h}=h\exp(\int_{\mu}^{\bar{\mu}}\frac{d\mu}{\mu}\gamma(\bar{g}(\mu))),\ \bar{m}=m\exp(\int_{\mu}^{\bar{\mu}}\frac{d\mu}{\mu}\eta(\bar{g}(\mu))),$$
$$\int_{g}^{\bar{g}}\frac{dg}{\beta(g)}=\ln\frac{\bar{\mu}}{\mu} \tag{28}$$

Explicit form of Generating function in the case of KNO scaling

Let us find generating function in the case of KNO scaling. From the definition of Generating function and using topological cross section from KNO, we find

$$F(h) = \sum_n h^n \frac{\sigma}{<n>} \Psi(\frac{n}{<n>}) = \frac{\sigma}{<n>} \sum \Psi(\frac{n}{<n>}) h^n$$
$$= \frac{\sigma}{<n>} \Psi(\frac{\delta}{<n>}) \frac{h^2}{1-h}, \ \delta \equiv h\frac{d}{dh}, \ q^\delta f(h) = f(qh), \tag{29}$$

No we can find more concrete form of the generation function, with the explicit form of KNO function,

$$(\frac{\delta}{<n>})^{c-1} \exp(-c\frac{\delta}{<n>}) \frac{h^2}{1-h} = (\frac{\delta}{<n>})^{c-1} \frac{q^2h^2}{1-qh}$$
$$= \frac{1}{<n>^{c-1}} \frac{1}{\Gamma(1-c)} \int_0^\infty \frac{dt}{t^c} \frac{q^2h^2e^{-2t}}{1-qhe^{-t}}, \tag{30}$$

so

$$F(h)_{KNO} = \frac{c^c}{\Gamma(c)} \frac{\sigma}{<n>^c} \frac{1}{\Gamma(1-c)} \int_0^\infty \frac{dt}{t^c} \frac{q^2h^2e^{-2t}}{1-qhe^{-t}},$$
$$q = \exp(-\frac{c}{<n>}) \tag{31}$$

Negative binomial distribution and KNO scaling

Let us consider negative binomial distribution (NBD) for normed topological cross sections

$$\frac{\sigma_n}{\sigma} = P(n) = \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}\left(\frac{k}{<n>}\right)^k\left(1+\frac{k}{<n>}\right)^{-(n+k)}, \tag{32}$$

where $k > 0$.

The generating function for NBD is

$$F(h) = \left(1+\frac{<n>}{k}(1-h)\right)^{-k} = \left(1+\frac{<n>}{k}\right)^{-k}(1-ah))^{-k},$$
$$a = \frac{<n>}{<n>+k}. \tag{33}$$

Indeed,

$$(1-ah))^{-k} = \frac{1}{\Gamma(k)}\int_0^\infty dtt^{k-1}e^{-t(1-ah)} = \frac{1}{\Gamma(k)}\int_0^\infty dtt^{k-1}e^{-t}\sum_0^\infty \frac{(tah)^n}{n!}$$
$$= \sum_0^\infty \frac{\Gamma(n+k)a^n}{\Gamma(k)n!}h^n,$$
$$P(n) = \left(1+\frac{<n>}{k}\right)^{-k}\frac{\Gamma(n+k)}{\Gamma(k)n!}\left(\frac{<n>}{<n>+k}\right)^n$$

$$
\begin{aligned}
&= \frac{k^k\Gamma(n+k)}{\Gamma(k)\Gamma(n+1)}(<n> +k)^{-(n+k)} <n>^n \\
&= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(\frac{k}{<n>})^k(1+\frac{k}{<n>})^{-(n+k)}
\end{aligned} \tag{34}
$$

Dispersion low for NBD

From the generating function we have

$$< n^2 >= (\frac{hd}{dh})^2 F(h)|_{h=1} = \frac{k+1}{k} < n >^2, \tag{35}$$

for dispersion we obtain

$$D = \sqrt{< n^2 > - < n >^2} = \frac{1}{\sqrt{k}} < n >, \tag{36}$$

so the dispertion low for KNO and NBD distributions are the same, with $c = k$. When we calculate normalized moments,

$$C_m = \frac{\Gamma(m+k)}{\Gamma(m)k^m}, \tag{37}$$

we fined that they are also same for KNO (see Apendix) and NBD distributions.

The moments coincides, but it not means that the KNO and NBD functions are the same. In the case of KNO we have integral formula for moments and for NBD we have summations for them. In that sense, NBD is good discrete approximation of the KNO, or KNO is good continual approximation for NBD.

The KNO as asymptotic NBD

Let us show that NBD is discrete distribution corresponding to the KNO scaling,

$$\lim_{<n>\to\infty} <n> P_n|_{\frac{n}{<n>}=z} = \Psi(z) \tag{38}$$

Indeed, using the following asymptotic formula

$$\Gamma(x+1) = x^x e^{-x}\sqrt{2\pi x}(1+\frac{1}{12x}+O(x^{-2})), \tag{39}$$

we find

$$\begin{aligned} <n> P_n &=<n> \frac{(n+k-1)^{n+k-1}e^{-(n+k-1)}}{\Gamma(k)n^n e^{-n}}\frac{k^k}{n^k} <n> z^k e^{-k\frac{n+k}{<n>}} \\ &= \frac{k^k}{\Gamma(k)} z^{k-1}e^{-kz} + O(1/<n>) \end{aligned} \tag{40}$$

We can calculate also $1/<n>$ correction term to the KNO from the NBD. The answer is

$$\Psi = \frac{k^k}{\Gamma(k)} z^{k-1}e^{-kz}(1+\frac{k^2}{2}(z-2+\frac{k-1}{kz})\frac{1}{<n>}) \tag{41}$$

This form coincides with corrected KNO for $c=k$ and $C_2^1 = 1$.

Appendix

Mellin transformation and $\Psi-$function

We take momentum or Mellin transform as

$$C_k = \int_0^\infty dz z^k \Psi(z) \tag{42}$$

and corresponding inverse momentum - Mellin transform as (is)

$$\Psi(z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dk z^{-k-1} C_k, \tag{43}$$

(indeed)

$$\begin{aligned}
&\Psi(z) = \frac{1}{2\pi i}\int_0^\infty du\Psi(u)\int_{-i\infty}^{i\infty}\frac{dk}{z}(\frac{u}{z})^k \\
&= \int_0^\infty du\Psi(u)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{dk}{z}e^{k\ln\frac{u}{z}} = \Psi(z), \\
&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dkze^{k\ln\frac{u}{z}} = z\delta\ln\frac{u}{z} = \delta(u-z);
\end{aligned} \tag{44}$$

$$C_k = \int_0^{\infty} dz z^k \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dk' z^{-k'-1} C_{k'}$$
$$\int_{-i\infty}^{i\infty} dk' \frac{1}{2\pi i} \int_0^{\infty} dz z^{k-k'-1} C_{k'} = C_k,$$
$$\frac{1}{2\pi i} \int_0^{\infty} dz z^{k-k'-1} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dt e^{t(k-k')} = \delta(k-k'). \tag{45}$$

Now we construct Ψ-function as inverse Mellin transform of her moments

$$C_{k+1} = (1 + k(C_2 - 1))C_k$$
$$= (1 + k(C_2 - 1))(1 + (k-1)(C_2 - 1))...C_2$$
$$= a^{-k}(k+a)(k-1+a)...(1+a) = \frac{\Gamma(k+a+1)}{\Gamma(a)a^{k+1}}$$
$$= \frac{1}{\Gamma(a)a^{k+1}} \int_0^{\infty} dt t^{k+a} e^{-t}, \; a = \frac{1}{C_2 - 1}. \tag{46}$$

$$\Psi(z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dk z^{-k-1} C_k$$
$$= \frac{1}{\Gamma(a)z} \int_0^{\infty} t^{a-1} e^{-t} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dk (\frac{t}{az})^k$$

$$= \frac{1}{\Gamma(a)z}\int_0^\infty dt t^{a-1}e^{-t}\delta(\ln\frac{t}{az}) = \frac{a^a}{\Gamma(a)}z^{a-1}e^{-az},$$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dk(\frac{t}{az})^k = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dk e^{\ln\frac{t}{az}} = \delta(\ln\frac{t}{az}) = az\delta(t-az),$$

$$\delta(f(x)) = \frac{1}{|f'(x_0)|}\delta(x-x_0) \tag{47}$$

Indeed,

$$\begin{aligned} C_k &= \int_0^\infty z^k\Psi(z)dz \\ &= \frac{a^a}{\Gamma(a)}\int_0^\infty z^{k+a-1}e^{-az}dz \\ &= \frac{a^a}{\Gamma(a)}\frac{\Gamma(k+a)}{a^{a+k}} = \frac{\Gamma(k+a)}{a^k\Gamma(a)} \end{aligned} \tag{48}$$

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