Continuous wavelet analysis of model heavy-ion data

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Motivation:

to study the multiparticle collective flows of secondary particles produced in ultrarelativistic heavy ion collisions

Plan of the talk:

1. Short introduction to the multiresolution (wavelet) analysis;
2. Which basis functions to use, which functions to decompose and why?
3. The results of the wavelet transform: the general mathematical formulas along with the graphic representations of some chosen examples. Comparisons of the analytical, numerical and MC calculations;
4. Correction of the wavelet spectra for finite detector efficiency;
5. Statistical errors of the wavelet spectra;
6. Summary.

Wavelet analysis ➡ designed to study processes on different scales

The continuous wavelet transform of analyzed function (or distribution) *f(x)* has the form:

$$W_{\Psi}(a,b) = C_{\Psi} \int_{-\infty}^{\infty} f(x) \Psi_{a,b}(x)\, dx, \quad (1)$$

where C_{Ψ} is a normalizing factor.

$$\Psi_{a,b}(x) = \Psi\left(\frac{x-b}{a}\right)$$

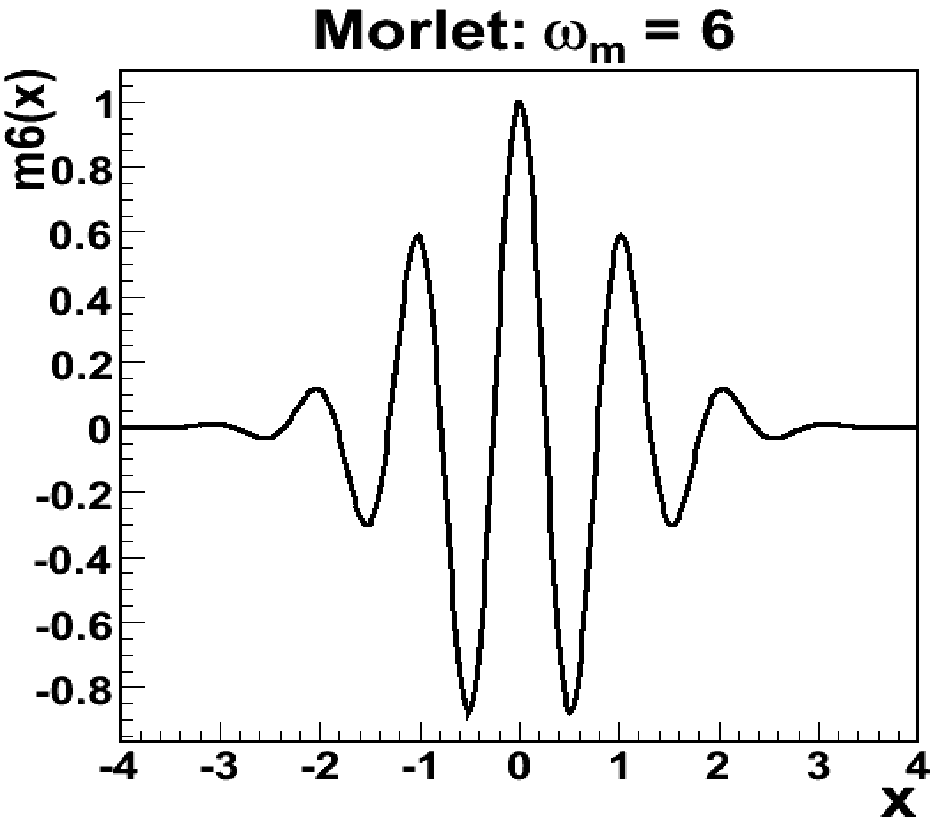
are shifted and/or dilated daughter functions generated from mother wavelet function $\Psi(x)$, b is a **translation** parameter and a is a **dilation** parameter or **scale**.

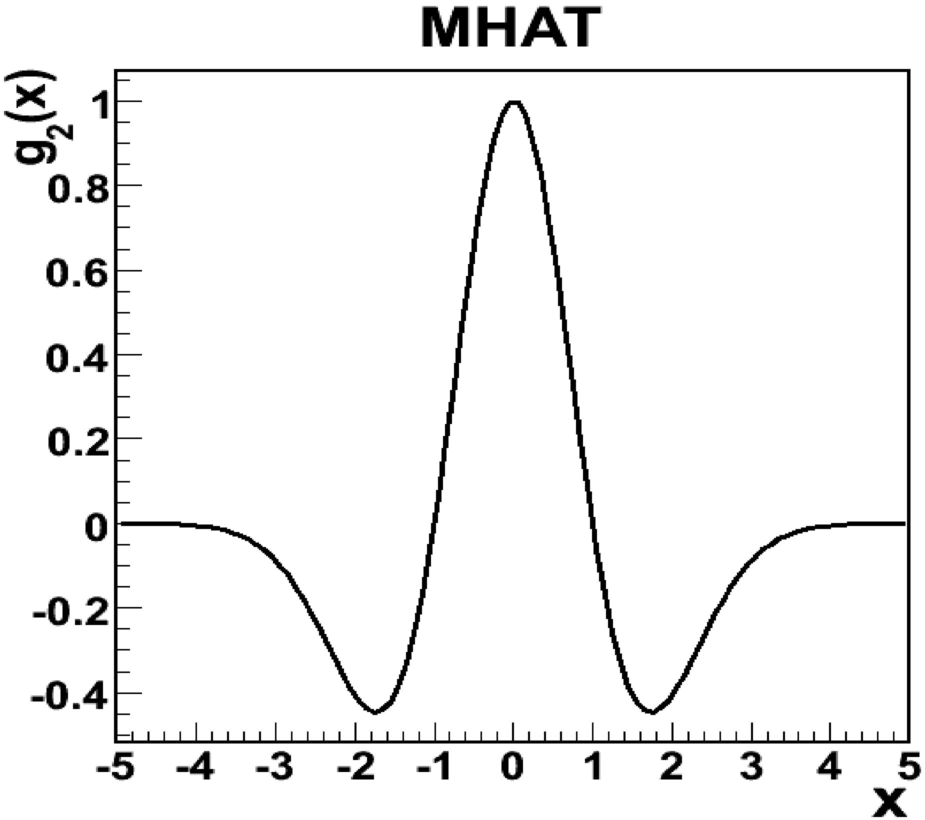
Coefficients $W_{\Psi}(a,b)$ are amplitudes of basis functions $\Psi_{a,b}$ in *f(x)*.

parameter ***a*** is related to signal **width**, ***b*** is related to its **position (phase)**

The employed basis functions Ψ:
(the plots are shown on the next slide)

1) Gaussian kernel (not a wavelet)
2) MHAT (or g_2) wavelet – the second derivative of Gaussian function
3) Morlet wavelet – cosine-modulated Gaussian function (the parameter is modulating frequency ω_m)





$$\Psi(x) = m6(x) = [\cos(\omega_m x) - \exp(-\omega_m^2/2)] \times$$
$$\times \exp(-x^2/2)$$

The constant term $\exp(-\omega_m^2/2)$ is added only to ensure the wavelet normalization will equal 0 when integrated from -∞ to ∞. It can be neglected for high frequencies.

$$\Psi(x) = g_2(x) = (1 - x^2)\exp(-x^2/2)$$

Gaussian

$$\Psi(x) = g(x) = \exp(-x^2/2)$$

Normalization factors C_Ψ of the employed basis functions:

$$\frac{1}{\pi^{1/4}\sqrt{a}}$$ for Gaussian,

$$\frac{2}{\sqrt{3}\,\pi^{1/4}\sqrt{a}}$$ for MHAT

and

$$\frac{1}{\pi^{1/4}\sqrt{0.5+1.5\exp(-\omega_m^2)-2\exp(-3\omega_m^2/4)}\sqrt{a}}$$ for Morlet.

They follow from the normalizing condition

$$C_\Psi^2 \int_{-\infty}^{\infty} \Psi_{a,b}^2(x)\,dx = 1.$$

The decomposed functions:

1) Gaussian as a prototype of jet structures (*the parameters: standard deviation σ, mean μ)*
2) cosine – to find out how the wavelet analysis is related to the “conventional” flow analysis *(the parameters: frequency ω, phase φ)*

Both the functions are normalized to unity since they are regarded probability density functions:

1) Gaussian by $1/(\sigma\sqrt{2\pi})$
2) (raised) cosine by

$$C_{\cos} = 1/\{[\sin(\omega x_{max} + \phi) - \sin(\omega x_{min} + \phi)]/\omega + C(x_{max} - x_{min})\}$$

where x_{max} and x_{min} are upper and lower bound of analyzed interval and *C* is a constant of uniform distribution on which the cosine is superimposed.

Analytical solutions of integral (1) when decomposing **Gaussian**:

for Gaussian:

$$W_g(a,b) = \frac{1}{\pi^{1/4}} \sqrt{\frac{a}{a^2+\sigma^2}} \underbrace{\exp\left[\frac{-(b-\mu)^2}{2(\sigma^2+a^2)}\right]}_{\text{Gaussian with sigma } \sqrt{\sigma^2+a^2}} \qquad (2)$$

Gaussian with sigma $\sqrt{\sigma^2+a^2}$

for MHAT

$$W_{g_2}(a,b) = \frac{2a^{5/2}}{\sqrt{3}\,\pi^{1/4}(\sigma^2+a^2)^{3/2}} \underbrace{\left[1-\frac{(b-\mu)^2}{\sigma^2+a^2}\right] \exp\left[\frac{-(b-\mu)^2}{2(\sigma^2+a^2)}\right]} \qquad (3)$$

MHAT with sigma $\sqrt{\sigma^2+a^2}$

At small scales: $\sigma^2+a^2 \to \sigma^2$ ⇒constant width; at large scales: $\sigma^2+a^2 \to a^2$ ⇒expanding width;

for Morlet

(4)

$$W_m(a,b)=\frac{1}{\pi^{1/4}\sqrt{\left[0.5+1.5\exp\left(-\omega_m^2\right)-2\exp\left(-0.75\,\omega_m^2\right)\right]}}\sqrt{\frac{a}{\sigma^2+a^2}}\exp\left[\frac{-\omega_m^2\sigma^2}{2\left(\sigma^2+a^2\right)}\right]\times$$

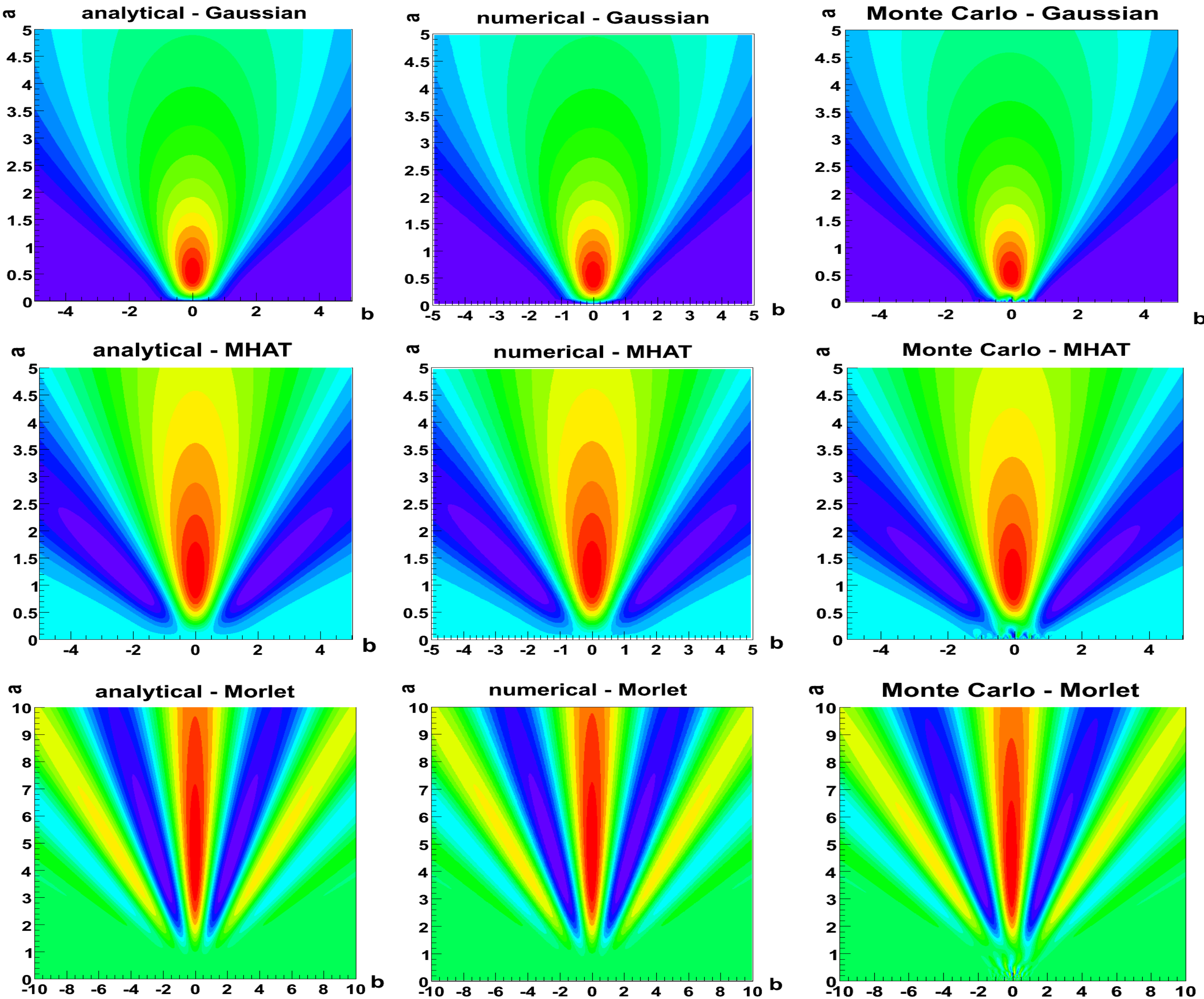
$$\times\underbrace{\left\{\cos\left[\frac{a}{\sigma^2+a^2}\omega_m\left(b-\mu\right)\right]-\exp\left[\frac{-\omega_m^2 a^2}{2\left(\sigma^2+a^2\right)}\right]\right\}\exp\left[\frac{-\left(b-\mu\right)^2}{2\left(\sigma^2+a^2\right)}\right]}$$

Morlet with sigma $\sqrt{\sigma^2+a^2}$ and frequency ω_m is modified by factor $a/(a^2+\sigma^2)$.

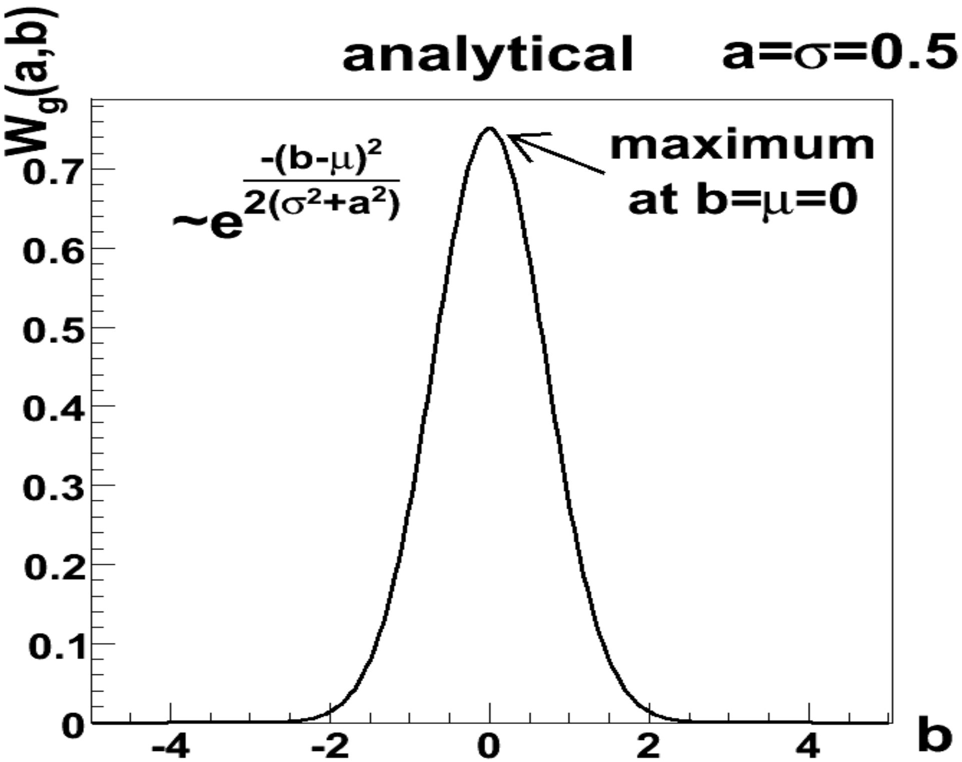
Summary: The phase dependences have always a shape of the employed basis functions with the same means but modified widths. Their amplitudes depend on scale *a* and on σ of the analyzed Gaussian. They go to 0 for either small or large scales.

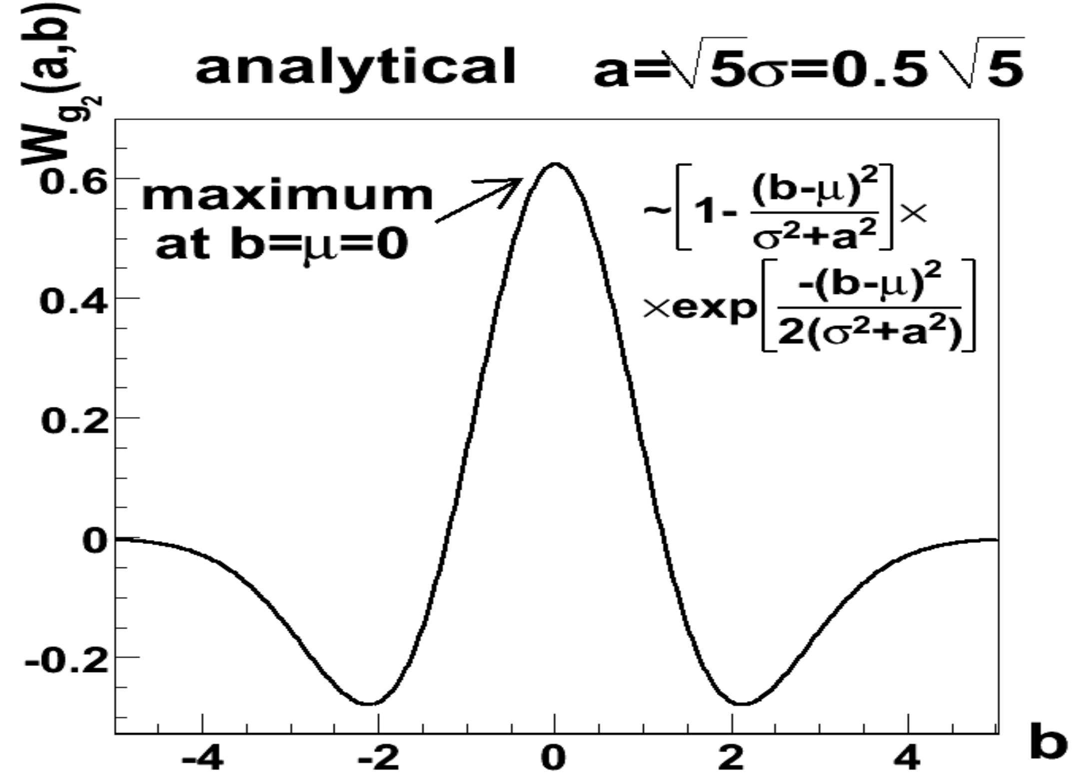
Example

The next slides present two-dimensional amplitude (i.e. $W_\psi(a,b)$) spectra of the Gaussian peak with σ=0.5 and mean μ=0 as well as the slices taken at their global maxima along *b* (phase) and along *a* (scale) axes. The analytical solutions are always compared with the corresponding numerical results and the Monte Carlo simulations when 100 Gaussian distributed values are generated. The Morlet frequency is set to $\omega_m = 6$ (although it can be arbitrary).

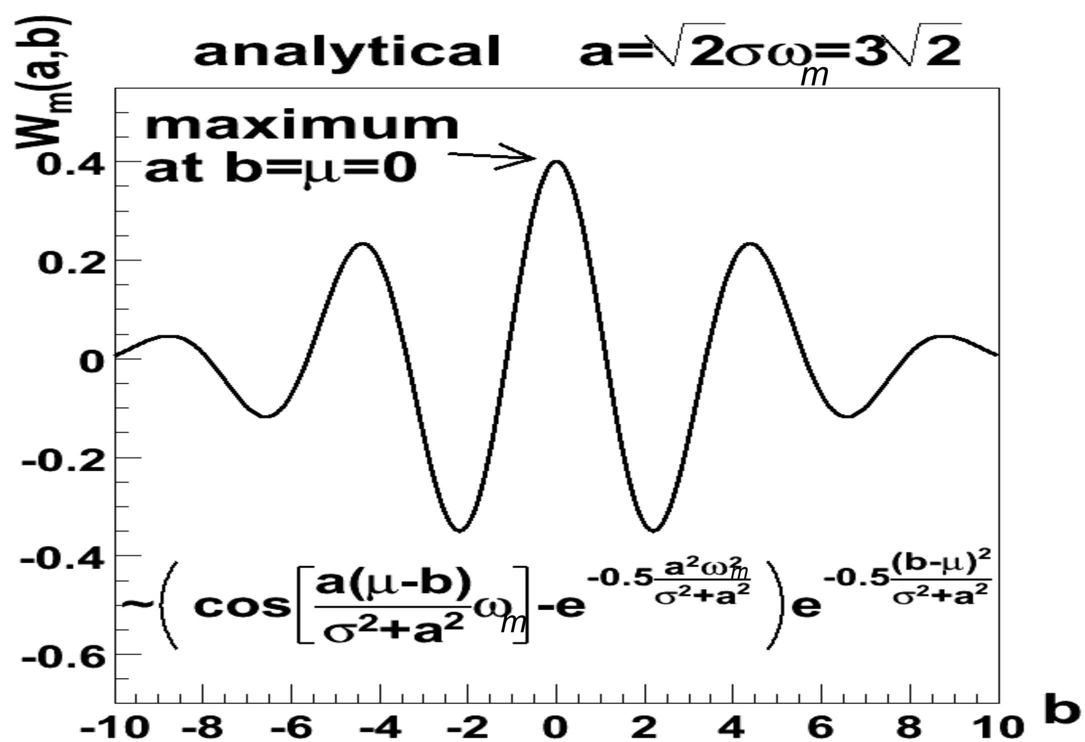


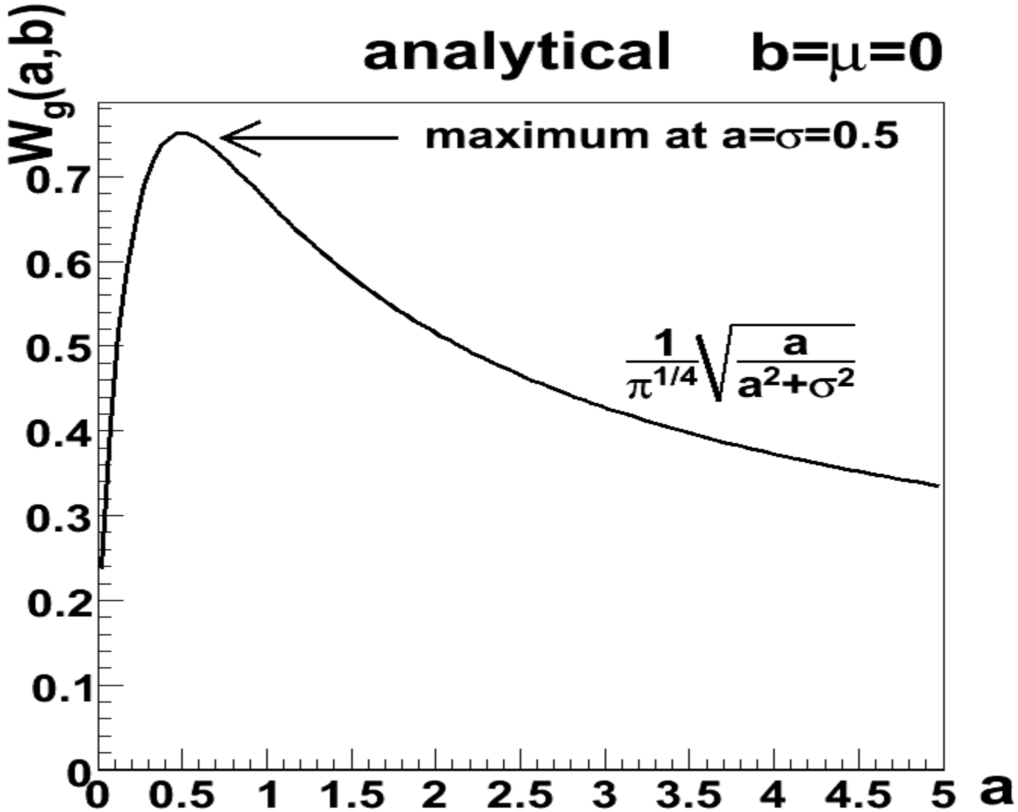
$W_\psi(a,b)$ spectra of the studied Gaussian peak. It is always represen-ted by the global maximum at phase b=μ=0 but at different scales, depend-ing on the used basis function.

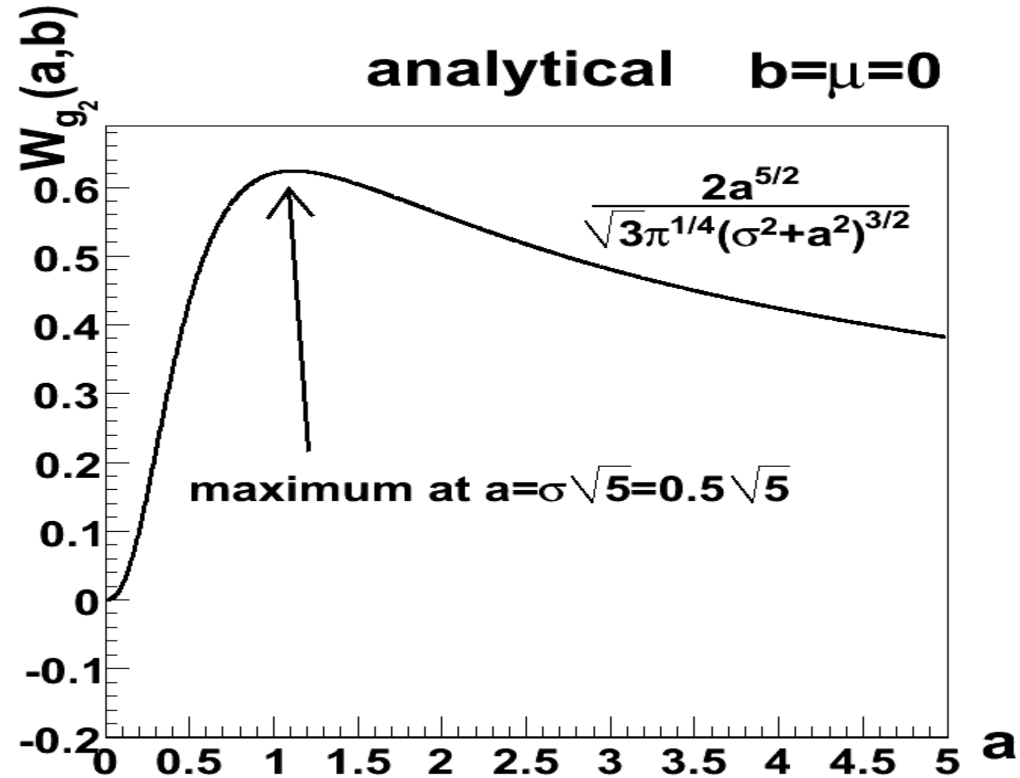




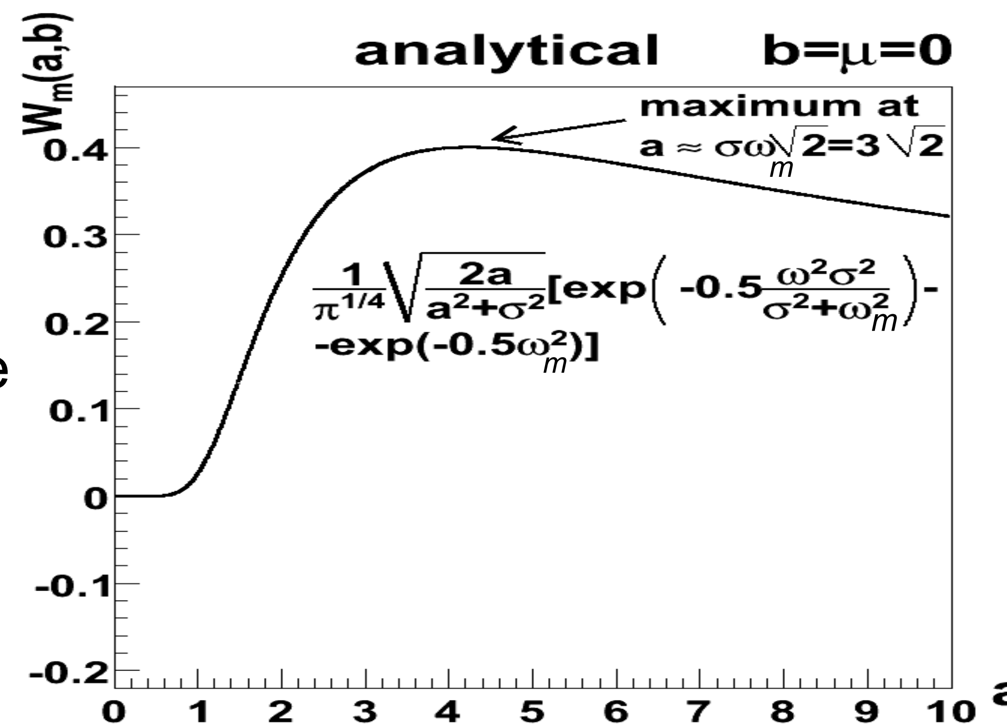
b slices at the maxima of $W_{\psi}(a,b)$ spectra for all the studied basis functions. The scales where the slices are done are shown in the Figures. The plots confirm the previous conclusions about the shapes of the phase dependences.







a slices at the maxima of $W_\psi(a,b)$ spectra for all the studied basis functions. The maxima imply the scales where the analyzed Gaussian is best visible, the widths of the spectra indicate the resolutions of the Gaussian σ.



Analytical solutions of integral (1) when **cosine** is decomposed:

for Gaussian:

$$W_g(a,b) = C_{\cos}\pi^{1/4}\sqrt{2a}\cos(\omega b+\phi)\exp(-0.5a^2\omega^2) \quad (5)$$

for MHAT

$$W_{g_2}(a,b) = C_{\cos}2\pi^{1/4}\sqrt{\frac{2\pi a}{3}}\cos(\omega b+\phi)a^2\omega^2\exp(-0.5a^2\omega^2) \quad (6)$$

for Morlet

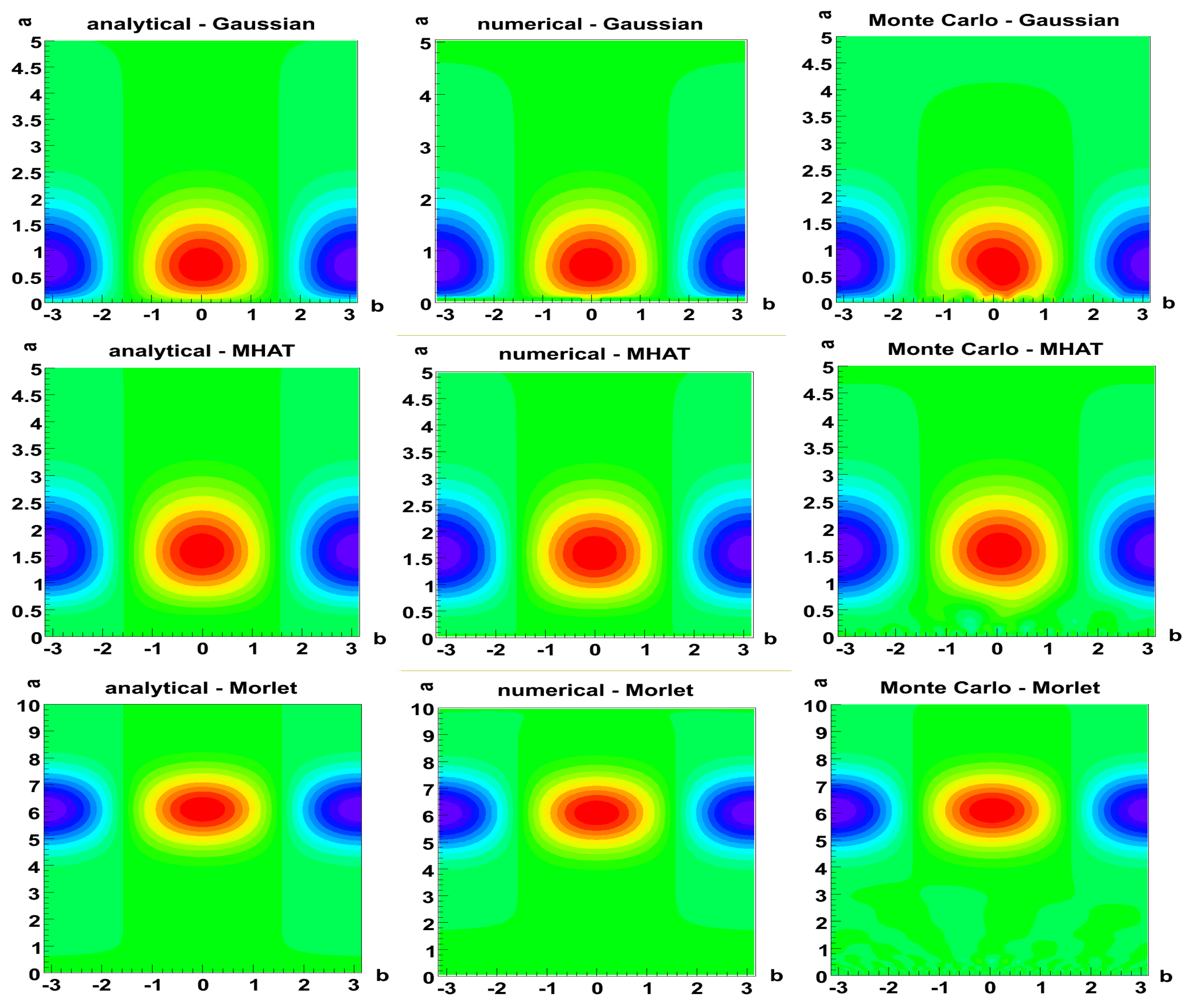
$$W_m(a,b) = \frac{C_{\cos}\pi^{1/4}}{\sqrt{[0.5+1.5\exp(-\omega_m^2)-2\exp(-0.75\omega_m^2)]}}\sqrt{\frac{a}{2}}\cos(\omega b+\phi)\times \quad (7)$$

$$\times\{\exp[-0.5(\omega_m-\omega a)^2]+\exp[-0.5(\omega_m+\omega a)^2]\}$$

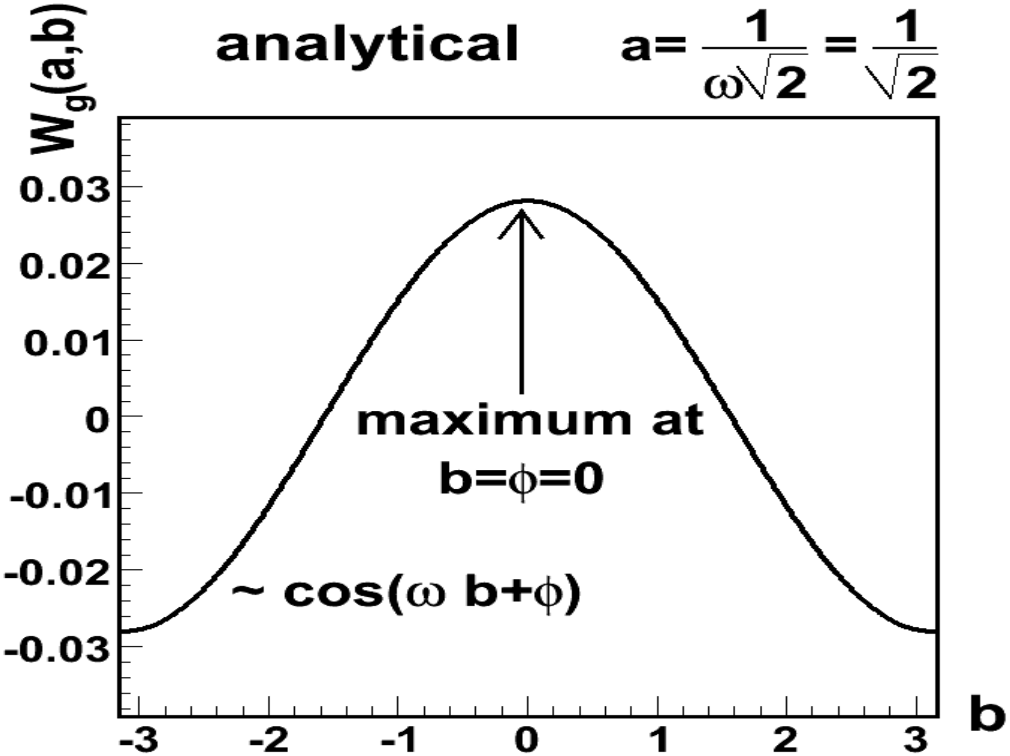
Summary: The phase dependence is always cosine with the original frequency and phase. The cosine amplitude depends on scale *a* and the frequency of the analyzed cosine. It goes to 0 for both small and large scales.

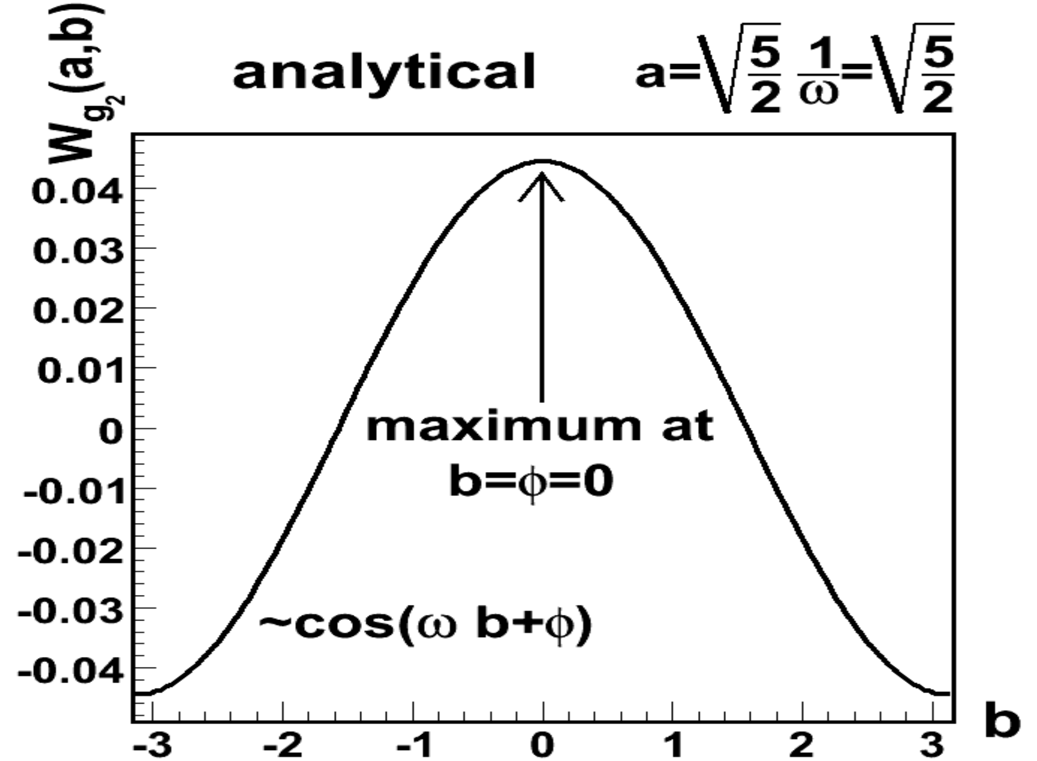
Example

The next slides present two-dimensional amplitude (i.e. $W_\Psi(a,b)$) spectra of the cosine function with frequency ω=1 and phase ϕ=0 along with the slices done at their maxima along *b* (phase) and along *a* (scale) axes. The analytical solutions are always compared with the corresponding numerical results and the Monte Carlo simulations when 500 raised cosine distributed values are generated. The Morlet frequency is chosen to be ω_m= 6.

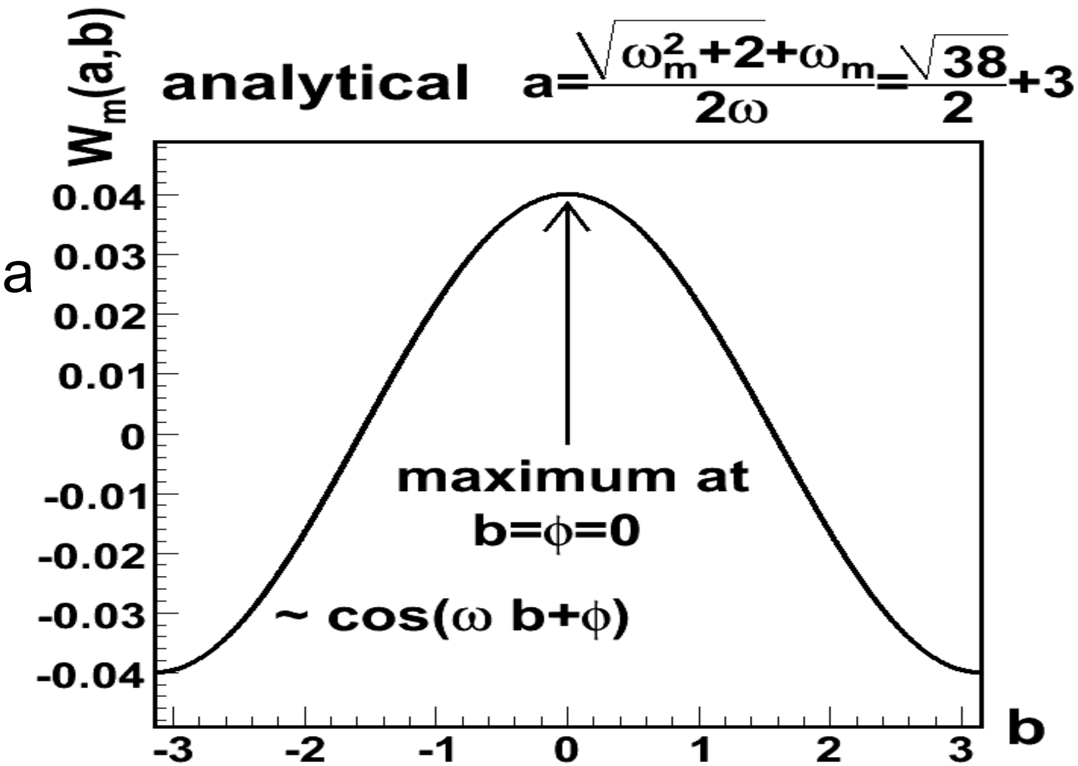


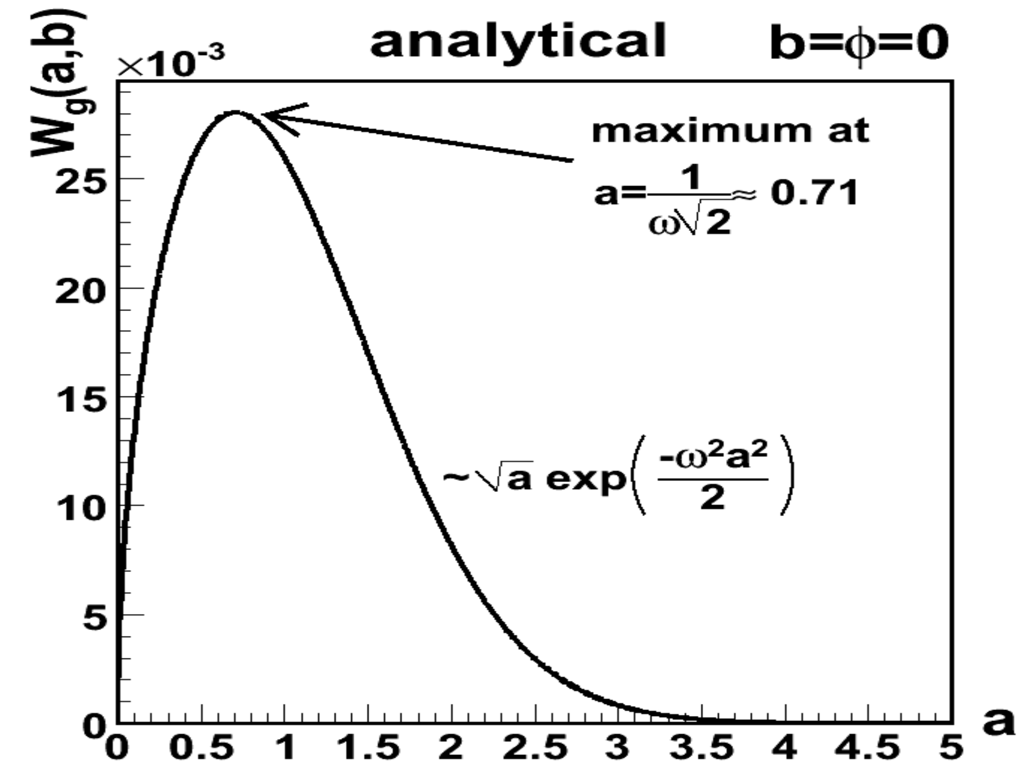
$W_{\Psi}(a,b)$ spectra of the studied cosine function. The cosine is always manifested by the maxima at phase $b=\varphi=0$ and minima at $b=\pm\pi$ but at different scales, depending on the used basis function.



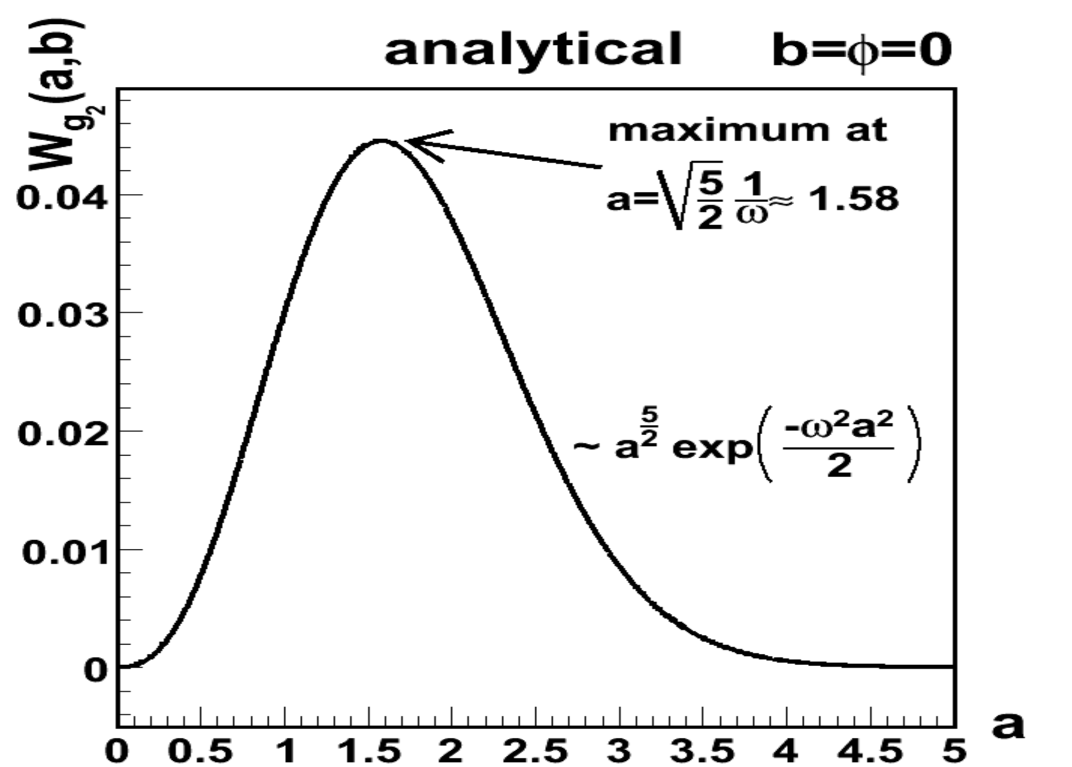


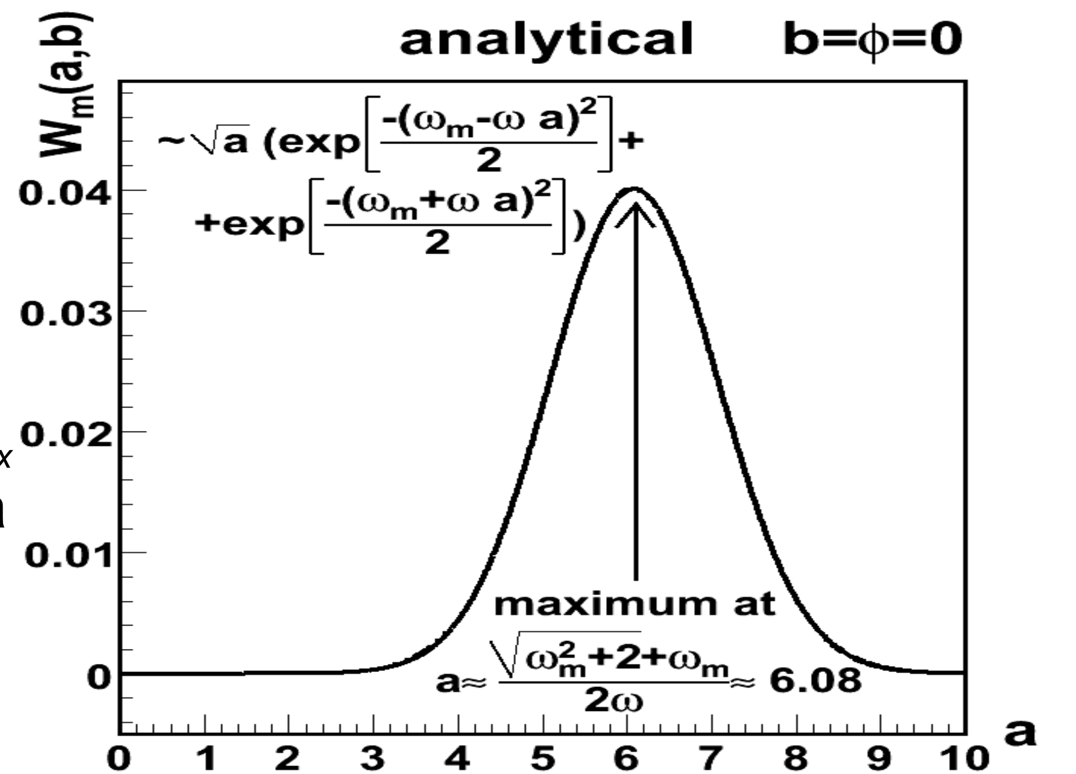
b slices at the maxima of $W_{\psi}(a,b)$ spectra for all the studied basis functions. The slices where the maxima are observed are done at the scales *a* which are displayed in the Figures. The plots are visual tests proving that the amplitude dependences on phase *b* have indeed expected cosine shapes.





a slices at the maxima of $W_\psi(a,b)$ spectra for all the studied basis functions. The plots indicate how is single Fourier frequency seen in scale space. Each ω is represented by spectrum of scales *a*. Roughly $\omega \leftrightarrow a_{max}$ (the position of maximum). The maxima for higher harmonics would appear closer to 0, since $a_{max}(\omega) \propto 1/\omega$ for all the tested basis functions.



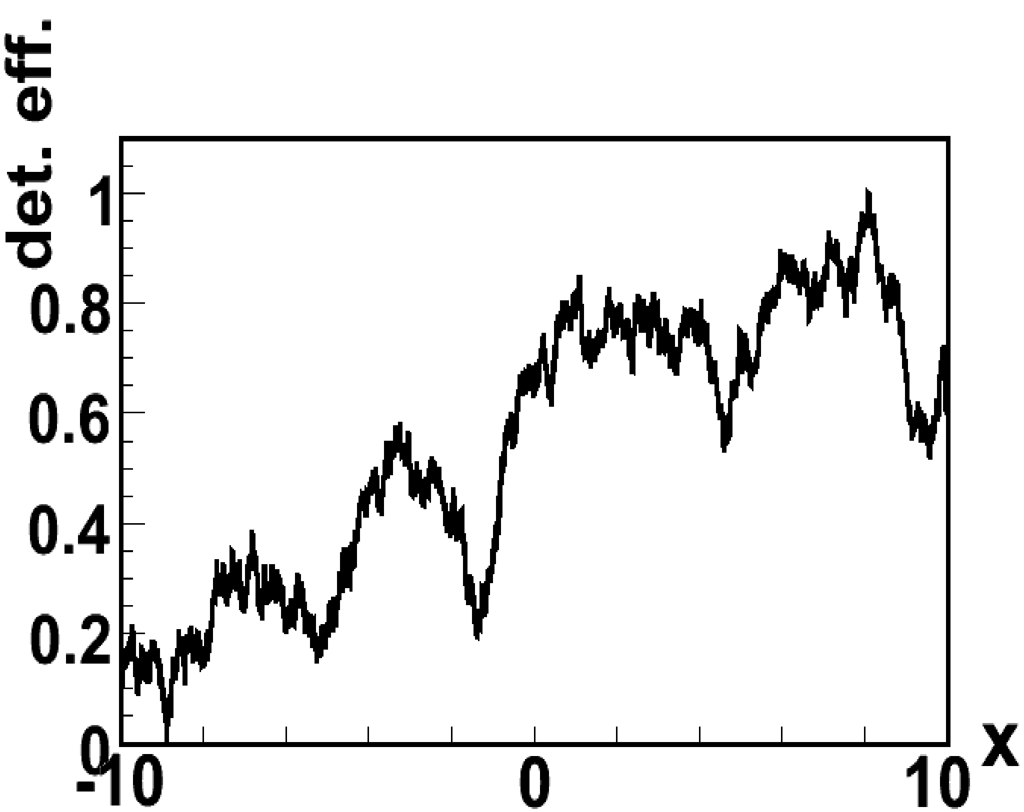


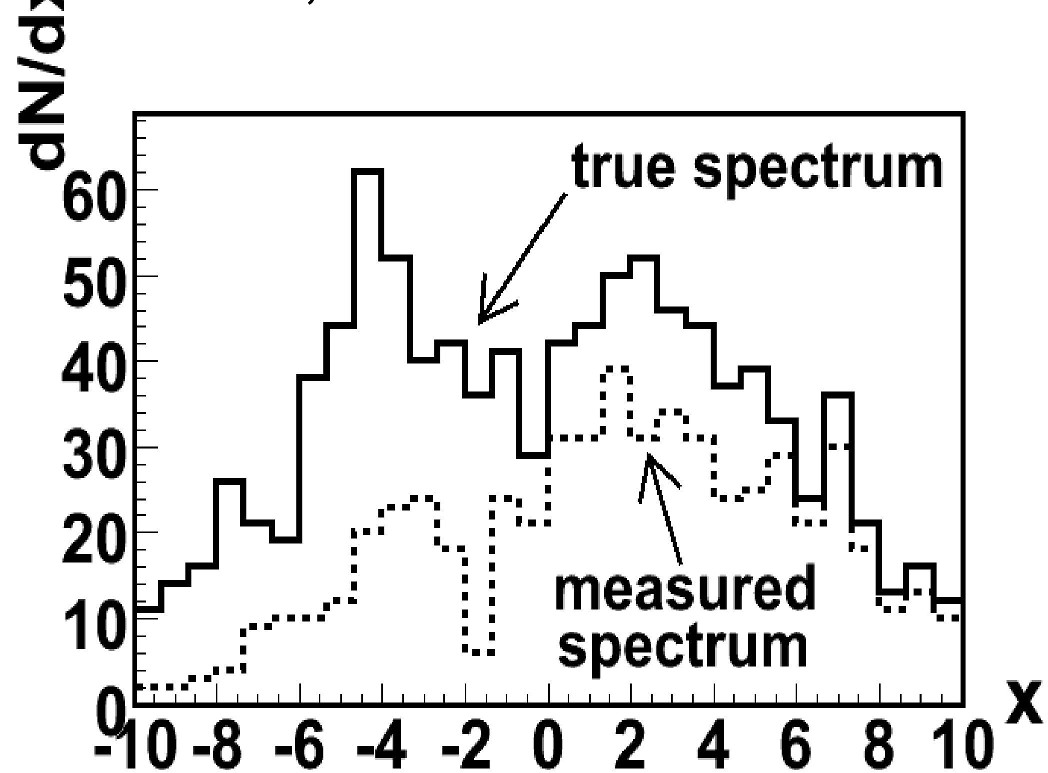
Correction of wavelet spectra for detector effects

How to correct wavelet spectra for limited detector efficiency?

Fig. left: The efficiency of some imaginary detector (in fact generated by random walk). The irregularities may distort wavelet spectra on all scales.

Fig. right: An impact of the same detector on the real (“true”) spectrum. The true spectrum consists of a couple of randomly distributed Gaussians with different σ's, the total statistics is 1000.





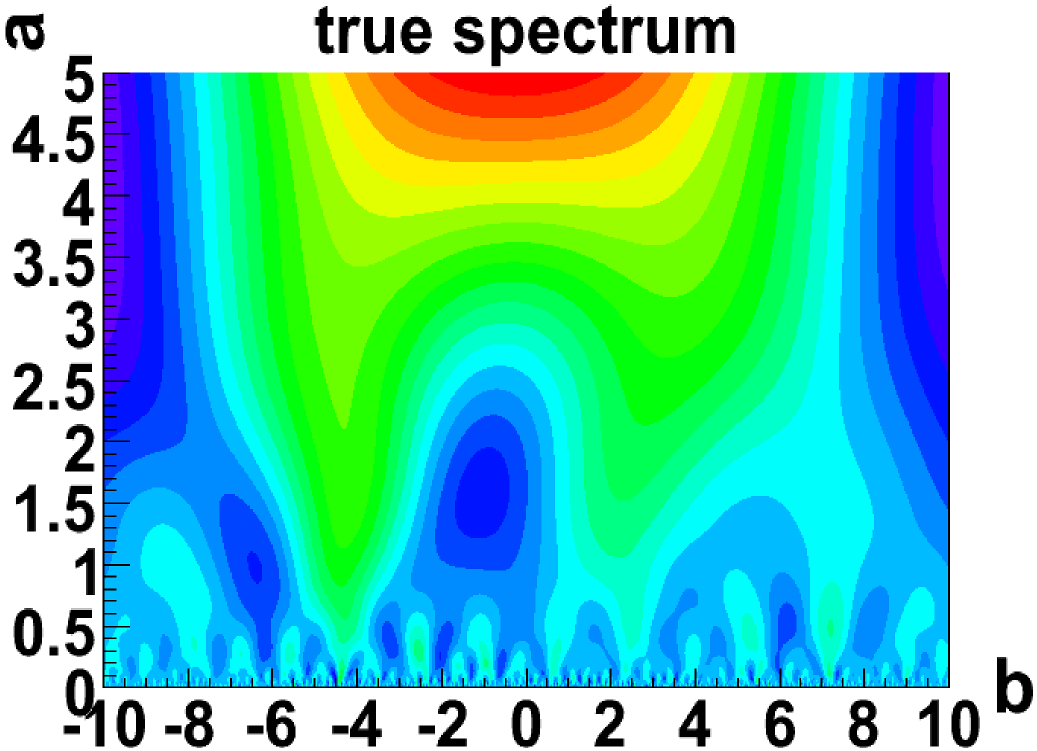
We come out from the identity: $W_{true} - W_{measured} = W_{true}^{uniform} - W_{measured}^{uniform},$ (8)

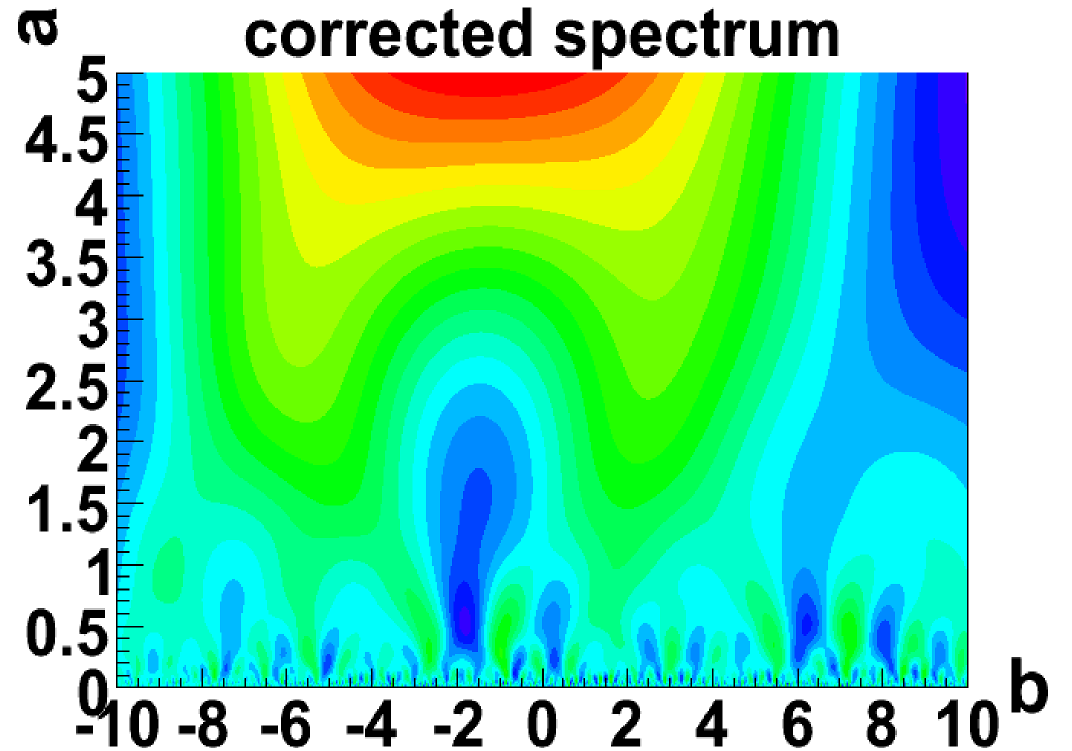
i.e. the difference between true and measured wavelet spectra of arbitrary distribution is same as the difference between true and measured wavelet spectra of uniform distribution.

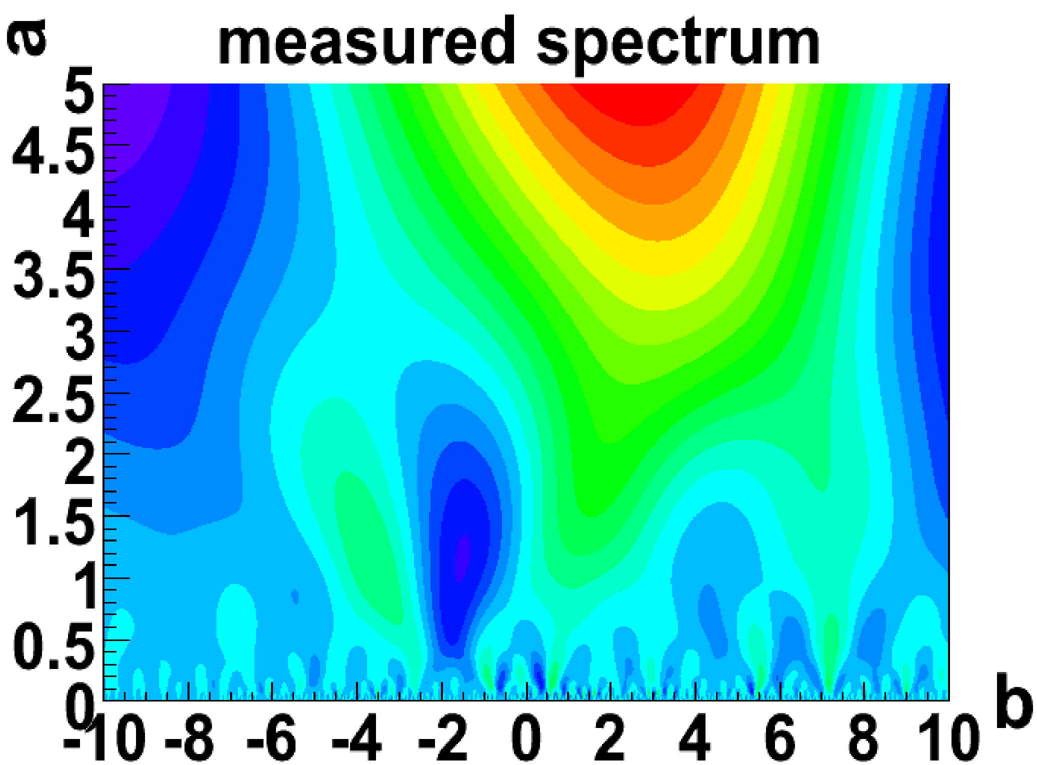
$W_{measured}^{uniform}$ is produced from large number of events which eventually create uniform true distribution even though the distributions of single events are not uniform.

In heavy ion collisions this condition is fulfilled for both (mid)rapidity and azimuthal distributions.

Remarks: 1) This correction assumes ideal detector resolution.
2) Detector efficiency should not change dramatically and erratically during the period when the event sample is measured.







The MHAT wavelet spectra of the previously shown event. Differences between the true and the corrected spectra seems larger on smaller scales which suggests they are primarily caused by statistical fluctuations.

➡ it is necessary to find out how the statistical deviations are propagated in the wavelet spectra;

If some distribution *f(x)* of *n* measurements $x_1, x_2, \ldots, x_n$ has the form:

$$f(x) = \frac{dN}{dx} = \frac{1}{n}\sum_{i=1}^{n} \delta(x - x_i), \qquad (9)$$

then its wavelet spectrum is $$W_\Psi(a,b) = \frac{C_\Psi}{n}\sum_{i=1}^{n} \Psi\left(\frac{x_i - b}{a}\right), \qquad (10)$$

i.e. integral in the formula (1) for the wavelet transform is replaced by sum.

Subsequently the errors of wavelet coefficients when calculated through error propagation are

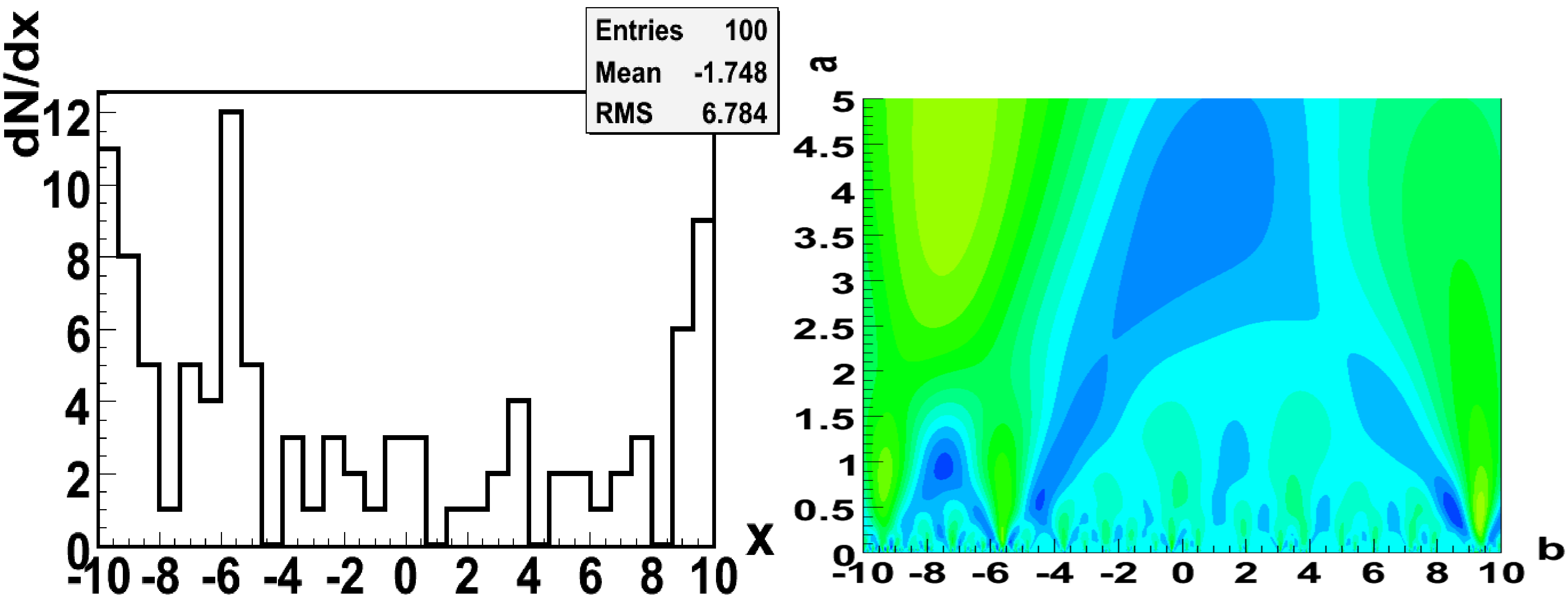
$$\sigma[W_\Psi(a,b)] = \frac{C_\Psi}{n}\sqrt{\sum_{i=1}^{n} \Psi^2\left(\frac{x_i - b}{a}\right)}, \qquad (11)$$

providing the input fluctuations are described by Poisson distribution.

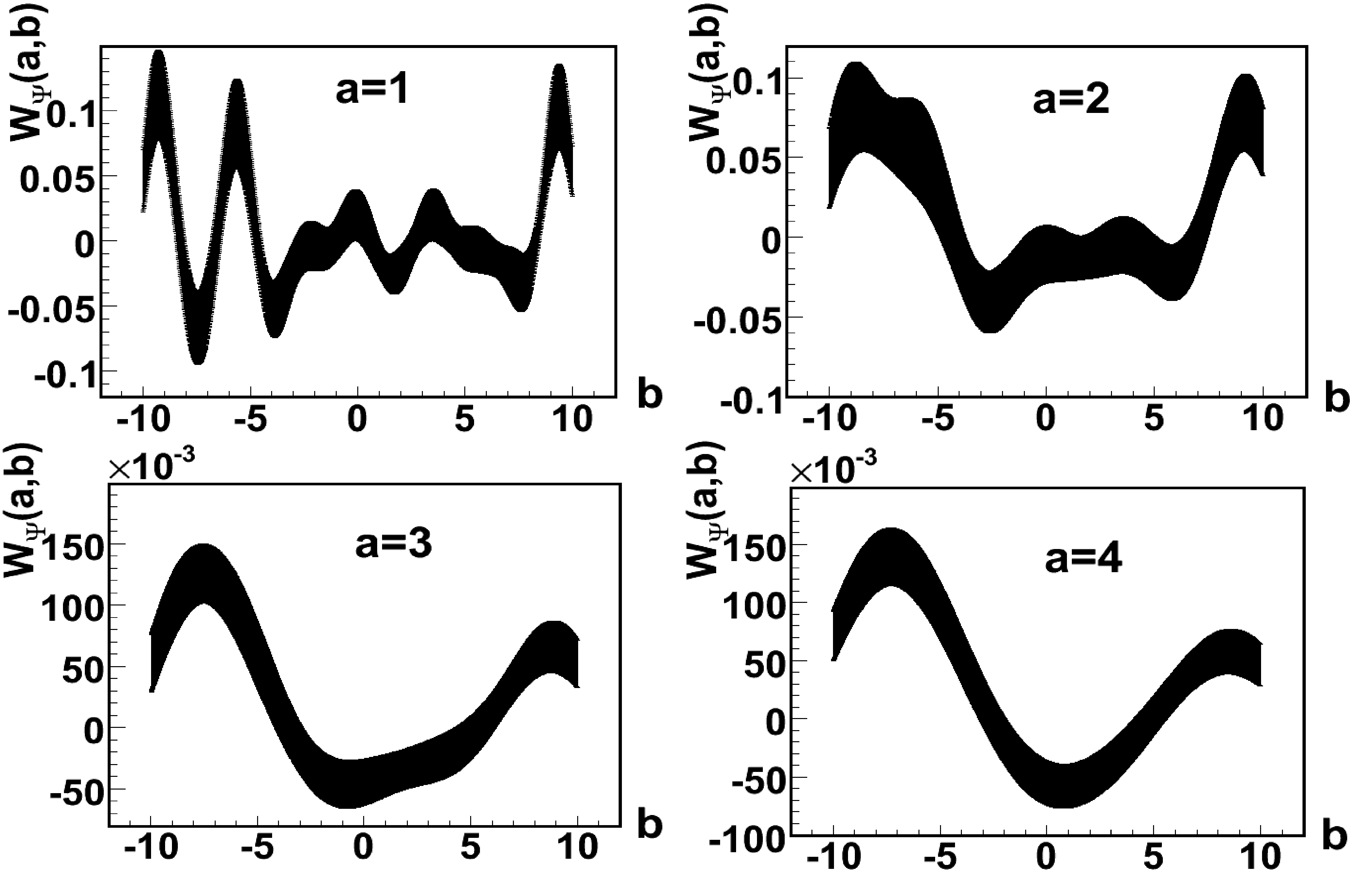
The resulting wavelet coefficients $W_{\psi}(a,b)$ are due to the <u>central limit theorem</u> Gaussian distributed which is valid mainly at larger scales.

Example: errors for the MC event shown below

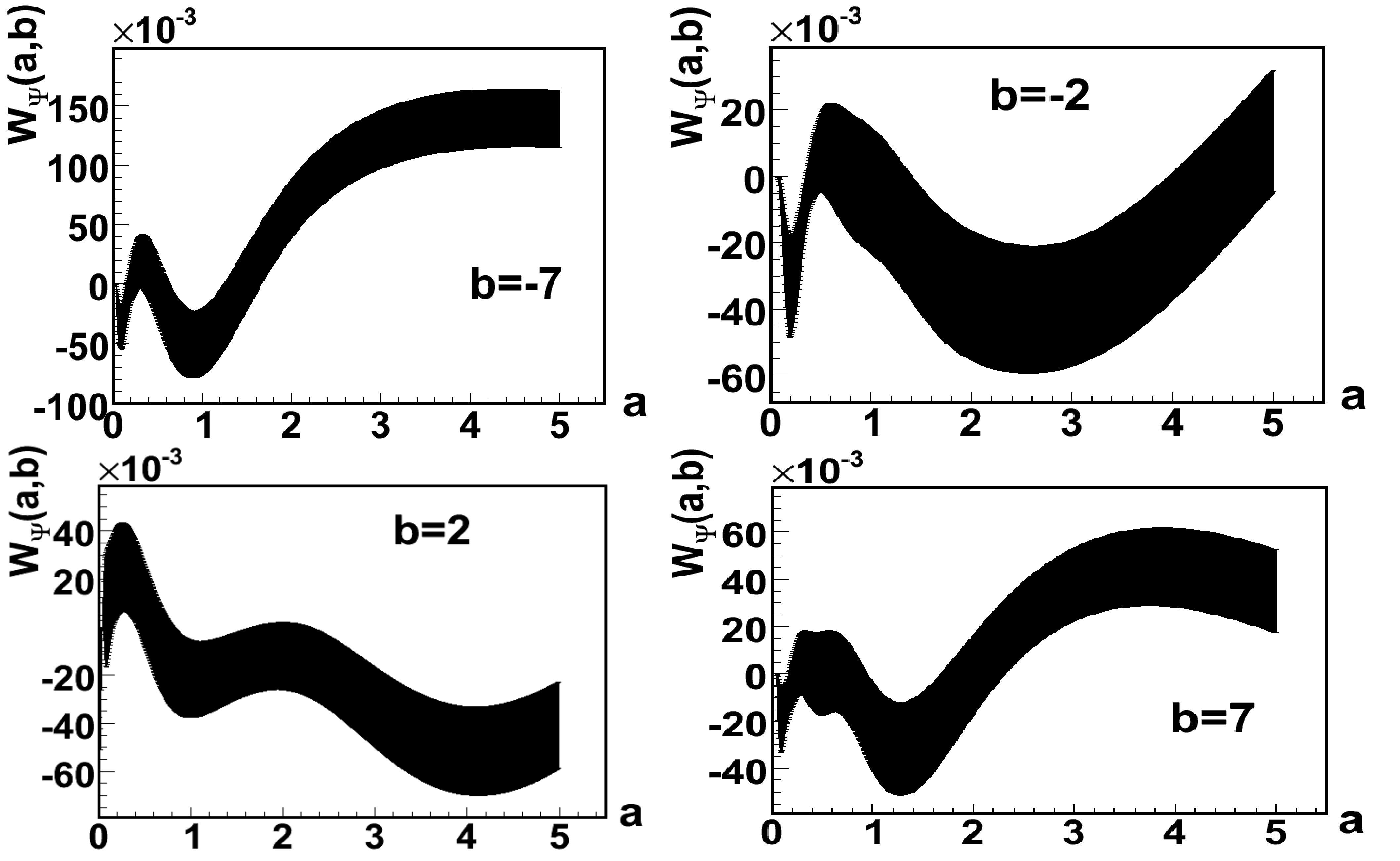
Figure: The distribution and the corresponding MHAT wavelet of MC event containing a few randomly distributed Gaussians with various σ's at the total statistics 100.



The phase wavelet spectra of the studied event. The plots are done at the scales ***a***=1, 2, 3, 4 and show the error corridors as well. Statistical significance of the observed structures has increasing trend when going to larger scales.



The scale wavelet spectra of the studied event along with the corresponding errors. The plots are done at the phases ***b***= -7, -2, 2, 7. The errors does not show significant scale dependence.



Knowing how to calculate the statistical errors, we may return to the previous problem.

1) We calculate directly errors for W_{true};

2) We evaluate errors of $W_{corrected}$ through error propagation using identity (8)

$$W_{corrected} = W_{measured} + W_{true}^{uniform} - W_{measured}^{uniform} \qquad (12)$$

providing the errors of the input wavelet spectra* are independent;

3) The spectra W_{true} and $W_{corrected}$ are compared if they are consistent within the resulting statistical errors. If yes, the correction works O.K..

The last step is not finished yet.

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* $W_{true}^{uniform}$ can be replaced by the analytical solution in order to eliminate this source of errors.

Summary and conclusions

1) Wavelet transform of Gaussian peak and cosine function was studied employing the various basis functions.
2) It is possible to determine the parameters of the decomposed functions from the amplitude (wavelet) spectra.
3) The analytical solutions are in very good agreement with the numerical calculations and the MC simulations.
4) It is presented how to correct the wavelet spectra for limited detector efficiency.
5) Propagation of statistical errors in the wavelet spectra is estimated.

Plan:

1) Scalograms (integrals of wavelet power spectra over a certain range of scales)
2) To estimate errors of parameters extracted from wavelet spectra