Duality and factorization theorem in QCD

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based on
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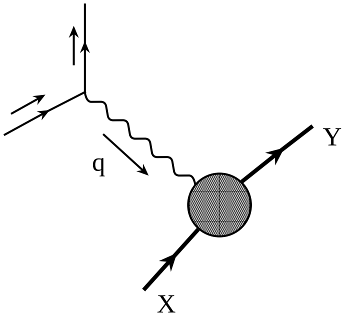
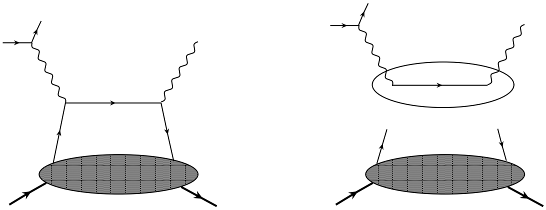
Hard reactions: DIS, DVCS, etc.

Hard reactions are reactions with a large transferred momentum, $-q^2 = Q^2 \to \infty$.

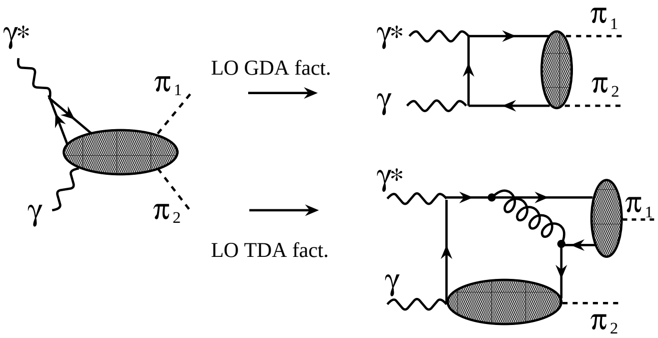


Factorization Theorem

The factorization theorem states that the dynamics of short and large distances can be separated out provided the transfer momentum is large.



Factorization in "s" – and "t" – channels: GDAs and TDAs



Factorization in overlap region: GDAs vs. TDAs

When both "s" and "t" are small compared to Q^2, both GDA- and TDA- schemes of factorization can be performed for $\gamma^*\gamma \to \pi\pi$.

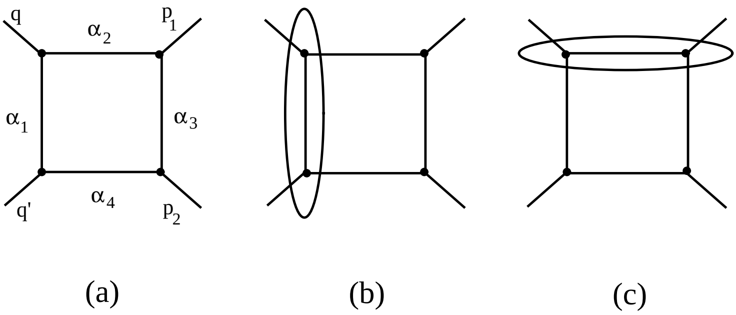
Therefore, a question now is:

ADDITIVITY: $\beta\,\text{GDA} + (1-\beta)\,\text{TDA}$

or

DUALITY: GDA = TDA ?

Duality in φ^3_E model



The most convenient way to analyze this four-particle amplitude in the φ^3_E model is to work within the α-representation [Radyushkin'96]. The reason is that within this representation

- the calculations can be performed in a systematic way, not only at one loop but also at higher-loop levels,
- the factorization of the process can be studied in detail, and
- one can study the spectral properties of the nonperturbative input (GPDs, GDAs) because these properties are insensitive to the numerators of the quark and gluon propagators, while complications owing to the spin structure do not affect them.

It is worth recalling in this context that the α-representation of Feynman diagrams is not only a schematic means for identifying relevant integration regions in a factorization procedure, but can serve to provide rigorous proofs of factorization theorems [Efremov-Radyushkin].

The contribution of the leading "box" diagram can be written as

$$\mathcal{A}(s,t,m^2) = -\frac{g^4}{16\pi^2}\int\limits_0^\infty \frac{\prod\limits_{i=1}^{4} d\alpha_i}{D^2} \exp\left[-\frac{1}{D}\left(Q^2\alpha_1\alpha_2 + s\alpha_2\alpha_4 + t\alpha_1\alpha_3 + m^2 D^2\right)\right],$$

where m^2 serves as a infrared (IR) regulator, $s > 0$, $t > 0$ are the Mandelstam variables in the Euclidean region, and $D = \sum\limits_{i=1}^{4}\alpha_i$.

Amplitude can be factorized under the proviso that, at least, $q^2 = Q^2$ is large compared to the mass scale m^2, which simulates here the typical scale of soft interactions. Recall that the factorization of the amplitude at the leading-twist level in the α-representation is equivalent to the calculation of the leading Q^2-asymptotics of the Laplace-type integral

$$F(\lambda) = \int_0^\infty d\alpha\, g(\alpha) \exp\left[-\lambda f(\alpha)\right] \approx \frac{g(0)}{\lambda\, f'(0)}$$

with a large and positive parameter λ and the function $f(\alpha)$ having a minimum at the point $\alpha = 0$.

With respect to the other two kinematic variables s and t, one can identify three distinct regimes of factorization:

(a) $s \ll Q^2$ while t is of order Q^2;

(b) $t \ll Q^2$ while s is of order Q^2;

(c) $s, \; t \ll Q^2$;

Regime (a): $s \ll Q^2$ while t is of order Q^2 (GDA)

The process is going through the s-channel, which would correspond to the GDA mechanism in QCD. In this regime, the main contribution in the integral in the amplitude (or, equivalently, the leading Q^2-asymptotics of the amplitude) arises from the integration over α_1 when $\alpha_1 \sim 0$:

$$\mathcal{A}^{\rm as}_{\rm GDA}(s,t,m^2) = -\frac{g^4}{16\pi^2}\int\limits_0^\infty \frac{d\alpha_2\, d\alpha_3\, d\alpha_4}{D^2}\, \exp\left(-s\frac{\alpha_2\alpha_4}{D} - m^2 D\right) \left[Q^2\frac{\alpha_2}{D} + t\frac{\alpha_3}{D} + m^2\right]^{-1} .$$

Then, by means of a rescaling:

$$\alpha_2 = \frac{x}{Q^2}, \quad \alpha_4 = \frac{y}{Q^2}, \quad \alpha_3 = \frac{z}{Q^2},$$

we are able to rewrite the asymptotic expression as

$$\mathcal{A}^{\rm as}_{\rm GDA}(s,t,m^2) = -\frac{g^4}{16\pi^2}\frac{1}{Q^4}\int\limits_0^\infty \frac{dxdydz}{x+y+z} \frac{\exp\left[-\frac{s}{Q^2}\frac{xy}{x+y+z} - \frac{m^2}{Q^2}(x+y+z)\right]}{x+zt/Q^2+(x+y+z)m^2/Q^2}.$$

Regime (b): $t \ll Q^2$ while s is of order Q^2 (TDA)

Here, we have to eliminate from the exponential in the amplitude the variables Q^2 and s, which are large. This can be achieved by integrating over the region $\alpha_2 \sim 0$, i.e., by cutting the line corresponding to this parameter. Performing similar manipulations as in regime (a), we find that the scalar TDA amplitude (associated with the factorization in the t-channel can be related to the scalar GDA via

$$\mathcal{A}^{\text{as}}_{\text{TDA}}(s,t,m^2) = \mathcal{A}^{\text{as}}_{\text{GDA}}(t,s,m^2)\,.$$

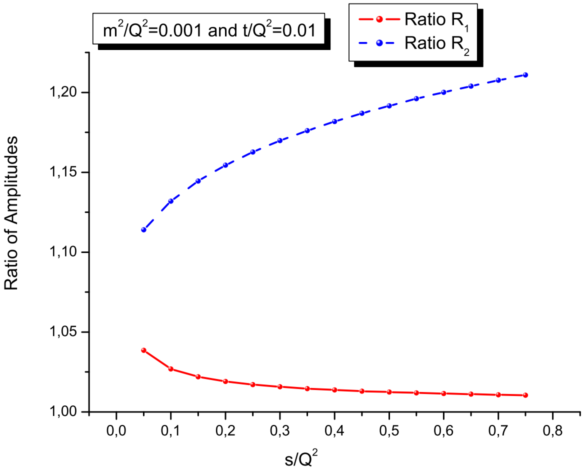
Regime (c): $t, s \ll Q^2$ (overlap)

The most important regime for our investigations on duality is when it happens that both variables s and t are simultaneously small compared to Q^2 (but still much larger than the soft scale m^2): $s,\ t \ll Q^2$. In this situation, there are two possibilities to extract the leading Q^2-asymptotics: we can either integrate over the region $\alpha_1 \sim 0$, or integrate instead over the region $\alpha_2 \sim 0$. These two possibilities correspond, respectively, to the GDA mechanism of factorization with the meson pair scattered at a small angle in its center-of-mass system or to the TDA mechanism of factorization.

We stress that we may face double counting, when naively adding these two contributions. We interpret such a behavior as a signal of duality between the GDA (s-channel) and the TDA (t-channel) factorization mechanisms.

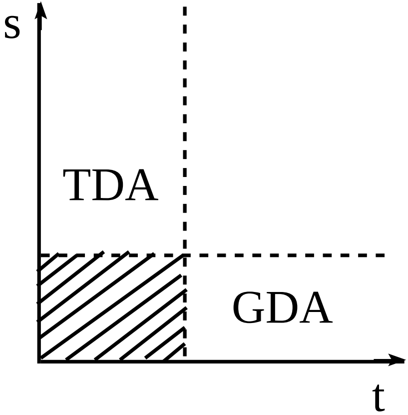
The simplest way to check the appearance of duality in the two different factorization schemes is to perform a numerical investigation of the exact and the asymptotic amplitudes expressed in the α-space representation. To this end, let us introduce the following ratios

$$R_1 = \frac{\mathcal{A}^{\rm as}_{\rm TDA}}{\mathcal{A}}\,, \quad R_2 = \frac{\mathcal{A}^{\rm as}_{\rm GDA}}{\mathcal{A}}\,.$$


m²/Q²=0.001 and t/Q²=0.01
Ratio R₁
Ratio R₂
Ratio of Amplitudes
1,00
1,05
1,10
1,15
1,20
0,0
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
s/Q²

One can see from this figure that in the region where s/Q^2 is rather small, i.e., in the range (0.01, 0.05), both asymptotic formulae are describing the exact amplitude with an accuracy of more than 90%. Because both ratios exceed unity, it becomes clear that additivity is excluded.

Thus, we may conclude that the duality between the TDA and the GDA factorization mechanisms takes place when both Mandelstam variables s/Q^2 and t/Q^2 belong to the rather broad interval (0.001, 0.7). In other words, in this interval the GDA and the TDA factorizations within the φ^3_E-model are equivalent to each other without a kinematical or dynamical prevalence of one over the other.


s
TDA
GDA
t

Duality in QCD

We now study the duality phenomenon in the case of real QCD. To this end, we consider the exclusive $\pi^+\pi^-$ production in a $\gamma_T \gamma_L^*$ collision, where the virtual photon with a large virtuality Q^2 is longitudinally polarized, whereas the other one is quasi real and transversely polarized. As already mentioned, the key feature of this process is that it exhibits two different kinds of factorization, based, respectively, on the GDA or the TDA mechanism, with a potential overlap of both mechanisms in the kinematical regime where t and s are small.

The $\gamma \to \pi^-$ matrix elements, entering the TDA-factorized amplitude, can be parameterized in the form

$$\langle \pi^-(p_2)|\bar{\psi}(-z/2)\gamma_\alpha[-z/2;z/2]\psi(z/2)|\gamma(q',\varepsilon')\rangle \stackrel{\mathcal{F}}{=} \frac{ie}{f_\pi}\epsilon_{\alpha\varepsilon'_T P\Delta_T} V_1(x,\xi,t)\,,$$

$$\langle \pi^-(p_2)|\bar{\psi}(-z/2)\gamma_\alpha\gamma_5[-z/2;z/2]\psi(z/2)|\gamma(q',\varepsilon')\rangle \stackrel{\mathcal{F}}{=} \frac{e}{f_\pi}\varepsilon'_T\cdot\Delta_T P_\alpha A_1(x,\xi,t)\,,$$

where $P = (p_2 + q')/2$, and $\Delta = p_2 - q'$.

To normalize the axial-vector TDA, A_1, we express A_1 in terms of the axial-vector form factor measured in the weak decay $\pi \longrightarrow l\nu_l\gamma$, i.e.,

$$\int_{-1}^{1} dx\, A_1(x,\,\xi,\,t) = 2\, f_\pi\, F_A(t)/m_\pi\,,$$

where $f_\pi = 0.131\,\text{GeV}$, $m_\pi = 0.140\,\text{GeV}$, and $F_A(0) \approx 0.012$.

The next objects of interest in our considerations are the helicity amplitudes that are obtained from the usual amplitudes after multiplying them by the photon polarization vectors:

$$\mathcal{A}_{(0,\pm)} = \varepsilon^{(0)}_{\mu} T^{\mu\nu}_{\gamma\gamma^*} \varepsilon'^{(\pm)}_{\nu},$$

$$\varepsilon'^{(\pm)} = \left(0, \frac{\mp 1}{\sqrt{2}}, \frac{+i}{\sqrt{2}}, 0\right), \quad \varepsilon^{(0)} = \left(\frac{|\vec{q}\,|}{\sqrt{q^2}}, 0, 0, \frac{q_0}{\sqrt{q^2}}\right).$$

Thus, the helicity amplitude associated with the TDA mechanism reads

$$\mathcal{A}^{\rm TDA}_{(0,j)} = \frac{\varepsilon'^{(j)} \cdot \Delta^T}{|\vec{q}\,|}\mathcal{F}^{\rm TDA},$$

$$\mathcal{F}^{\rm TDA} = [4\,\pi\,\alpha_s(Q^2)]\frac{C_F}{2\,N_c}\int\limits_0^1 dy\,\frac{\phi_\pi(y)}{y\bar{y}}$$

$$\int\limits_{-1}^{1} dx\,A_1(x,\xi,t)\left(\frac{e_u}{\xi-x}-\frac{e_d}{\xi+x}\right),$$

where $\xi^{-1} = 1 + 2W^2/Q^2$ and where we have employed the 1-loop $\alpha_s(Q^2)$ in the $\overline{\rm MS}$-scheme with $\Lambda_{\rm QCD} = 0.312$ GeV for $N_f = 3$.

On the other hand, the helicity amplitude which includes the twist-3 GDA can be written as

$$\mathcal{A}^{\rm GDA}_{(0,j)} = \frac{\varepsilon^{\prime\,(j)} \cdot \Delta^T}{|\vec{q}\,|}\mathcal{F}^{\rm GDA},$$

$$\mathcal{F}^{\rm GDA} = 2\frac{W^2+Q^2}{Q^2}(e_u^2+e_d^2)$$

$$\int_0^1 dy\,\partial_\zeta \Phi_1(y,\zeta,W^2)\left(\frac{\ln \bar{y}}{y} - \frac{\ln y}{\bar{y}}\right),$$

where $2\zeta - 1 = \beta \cos\theta^{\pi}_{cm}$ and $\cos\theta^{\pi}_{cm} = 2t/(W^2+Q^2) - 1$ and the partial derivative is defined by $\partial_\zeta = \partial/\partial(2\zeta - 1)$.

Modeling of non-perturbative objects: TDAs

We assume a factorizing ansatz for the t-dependence of the TDAs and write

$$A_1(x,\xi,t) = 2\,\frac{f_\pi}{m_\pi}\,F_A(t)A_1(x,\xi)\,,$$

where the t-independent function $A_1(x,\xi)$ is normalized to unity:

$$\int_{-1}^{1} dx A_1(x,\xi) = 1\,.$$

To satisfy this condition, we introduce a TDA defined by

$$A_1(x,1) = \frac{A_1^{\text{non-norm}}(x,1)}{\mathcal{N}},$$

$$\mathcal{N} = \int_{-1}^{1} dx A_1^{\text{non-norm}}(x,1)\,.$$

Using E.O.M., one can get the following relations:

$$A_1(x,1) \sim e\chi\langle\bar{\psi}\psi\rangle\varphi_\gamma(\frac{1+x}{2}).$$

We now focus on the discussion of the t-independent TDAs. Since we are mainly interested in TDAs in the Efremov-Radyushkin-Brodsky-Lepage (ERBL) region $\xi = 1$, it is instructive to choose the following parametrization

$$A_1^{\text{non-norm}}(x,1) = (1-x^2)\Big(1 + a_1 C_1^{(3/2)}(x) + a_2 C_2^{(3/2)}(x) + a_4 C_4^{(3/2)}(x)\Big),$$

where a_1, a_2, a_4 are free adjustable parameters, encoding nonperturbative input, and the standard notations for Gegenbauer polynomials are used.

Values of parameters

One appreciates that the TDA in this form amounts to summing a D-term, the term with the coefficient a_1, and meson-DA-like contributions. Indeed, would we eliminate the term with a_1, we would obtain the standard parametrization for a meson DA. On the other hand, keeping only the term with a_1, would reproduce the parametrization for the D-term. Therefore, for our analysis, we suppose that $a_1 \equiv d_0$, which is equal to -0.5 in lattice simulations.

With respect to the parameters a_2 and a_4, we allow them to vary in quite broad intervals, notably, $a_2 \in [0.3, 0.6]$ and $a_4 \in [0.4, 0.8]$, that would cover vector-meson DAs with very different profiles at a normalization scale $\mu^2 \sim 1\,\text{GeV}^2$.

Modeling of non-perturbative objects: GDAs

As regards the function $\Phi_1(z,\zeta)$, which denotes the corresponding GDA, we take recourse to the following model

$$\Phi_1(z,\zeta,W^2) = 9N_f\, z\bar{z}(2z-1)\Big(\tilde{B}_{10}(W^2)e^{i\delta_0(W^2)} + \tilde{B}_{12}(W^2)e^{i\delta_2(W^2)}P_2(\cos\theta_\pi)\Big),$$

where the phase shift of the $\pi\pi$ scattering is defined by $\delta_0(W_0 = 0.8) \approx \frac{\pi}{2}$ and $\delta_2(W_0 = 0.8) \approx 0.03\pi$.

Function $\tilde{B}_{10}$

The function $\tilde{B}_{10}$ corresponding to $L = 0$ does not contribute to Φ_1 and, therefore, can be discarded.

Function $\tilde{B}_{12}$

Since $\tilde{B}_{12}$ corresponds to the two-pion state with $L = 2$ and our GDAs probe only the isoscalar channel, we derive for $\tilde{B}_{12}$ at W^2 in the region of the f_2-meson $[I^G(J^{PC}) = 0^+(2^{++})]$ mass the following expression:

$$\tilde{B}_{12}(W^2) = \frac{10}{9}\frac{g_{f_2\pi\pi}\, f_{f_2}\, M_{f_2}^3\, \Gamma_{f_2}}{(M_{f_2}^2 - W^2)^2 + \Gamma_{f_2}^2 M_{f_2}^2}\,,$$

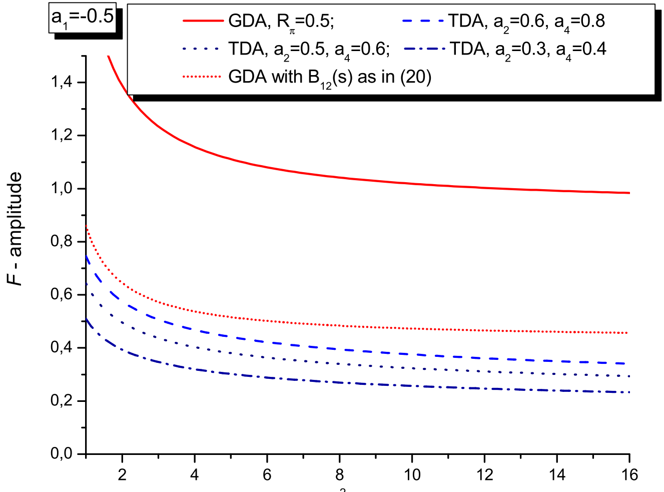
where $f_{f_2} = 0.056\,\text{GeV}$, $M_{f_2} = 1.275\,\text{GeV}$, $\Gamma_{f_2} = 0.185\,\text{GeV}$, and the decay constant $g_{f_2\pi\pi}$ is defined by

$$g_{f_2\pi\pi} = \sqrt{\frac{24\pi}{M_{f_2}^3}\,\Gamma(f_2 \to \pi\pi)}, \quad \Gamma(f_2 \to \pi\pi) = 0.85\Gamma_{f_2}\,.$$

In addition, we also model the function $\tilde{B}_{12}$ with the help of the most simple ansatz

$$\tilde{B}_{12}(0) = \beta^2 \frac{10}{9N_f} R_\pi \, ,$$

where R_π denotes the fraction of momentum carried by the quarks and antiquarks in the pion.


a1=-0.5
GDA, Rπ=0.5;
TDA, a2=0.6, a4=0.8
TDA, a2=0.5, a4=0.6;
TDA, a2=0.3, a4=0.4
GDA with B12(s) as in (20)
F - amplitude
0,0
0,2
0,4
0,6
0,8
1,0
1,2
1,4
2
4
6
8
10
12
14
16

SSA and Duality [cf. Teryaev's talk]

$$\mathcal{A}^{\text{SSA}} = \frac{d\sigma_{\rightarrow} - d\sigma_{\leftarrow}}{d\sigma_{\rightarrow} + d\sigma_{\leftarrow}} = \frac{\text{Im}[\,\rho_k^{(+,0)}\,]\,\text{Im}\Big[\overset{*}{\mathcal{A}}_{(+,+)}\;\mathcal{A}_{(0,+)}\Big]}{4\,\rho_k^{(+,+)}\;\big|\mathcal{A}_{(+,+)}\big|^2}\;.$$

where

$$\rho_k^{(i,i')} \overset{\text{def}}{=} [\,\overset{*}{\varepsilon}{}^{(i)} \cdot \mathcal{L}(k_1,k_2)\,]\,[\,\varepsilon^{(i')} \cdot \mathcal{L}^{+}(k_1,k_2)\,]\;,$$

$$\mathcal{L}_\alpha(k_1,k_2) \overset{\text{def}}{=} \bar{u}(k_2)\gamma_\alpha u(k_1)$$

Conclusions

- ★ We have shown that when it happens that both Mandelstam variables s and t are much less than the large momentum scale Q^2, with the variables s/Q^2 and t/Q^2 varying in the interval (0.001, 0.7), then the TDA and the GDA factorization mechanisms are equivalent to each other and operate in parallel.
- ★ We demonstrated that duality may serve as a tool for selecting suitable models for the non-perturbative ingredients of QCD factorization of various exclusive amplitudes.
- ★ We also observed that twist-3 GDAs appear to be dual to the convolutions of leading-twist TDAs and DAs, multiplied by a QCD effective coupling.